WELL-POSEDNESS FOR COMPRESSIBLE MHD SYSTEM WITH HIGHLY OSCILLATING INITIAL DATA

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ABSTRACT. In this paper, we transform compressible MHD system written in Euler coordinate to Lagrange coordinate in critical Besov space. Then we construct unique local solutions for compressible MHD system. Our results improve the range of Lebesgue exponent in Besov space from $[2, N)$ to $[2, 2N)$ with N stands for dimension. In addition, we give a lower bound for the maximal existence time which is important for our construction of global solutions. Based on the local solution, we obtain a unique global solution with high oscillating initial velocity and density by using effective viscous flux and Hoff's energy methods to explore the structure of compressible MHD system.

1. INTRODUCTION

Magnetohydrodynamics (MHD) is concerned with the interaction between fluid flow and magnetic field. The governing equations of isentropic compressible MHD system has the form

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) \\ \qquad\qquad = B \cdot \nabla B - \frac{1}{2}\nabla(|B|^2) + \mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \\ \partial_t B + (\operatorname{div} u)B + u \cdot \nabla B - B \cdot \nabla u = \nu \Delta B, \\ \operatorname{div} B = 0, \\ (\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0), \quad \operatorname{div} B_0 = 0, \end{cases}$$

where $\rho = \rho(t, x)$ denotes the density, $x \in \mathbb{R}^N$, $t > 0$, $u \in \mathbb{R}^N$ is the velocity of the flow, and $B \in \mathbb{R}^N$ stands for the magnetic field. The constants μ and λ are the shear and bulk viscosity coefficients of the flow, respectively, satisfying $\mu > 0$ and $\lambda + 2\mu > 0$, the constant $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. $P(\rho)$ is the scalar pressure function satisfying $P'(\bar{\rho}) > 0$ where $\bar{\rho}$ is the equilibrium state of the density.

MHD system has been studied by many authors for its physical importance and mathematical challenges. X. Hu and D. Wang constructed the global weak solutions with large initial data in a series of papers [7, 8, 9]. In addition, they also studied the important low march number limit problem in their papers. In 2012, A. Suen and D. Hoff [14] extends small energy weak solution theory about Navier-Stokes system to MHD case. Then global weak solutions with initial data contain vacuum also be constructed by many authors in [17, 18].

Besides the studies about weak solutions mentioned above, there are also a lot of investigations about classical and strong solutions. In 1984, S. Kawashima studied a very general symmetric hyperbolic-parabolic system which incorporate MHD system as a special one in his PhD thesis[28]. He constructed the global solution for MHD system when the initial data is small in $H^4(\mathbb{R}^3)$. After S. Kawashima's investigation, recently, many authors [20, 21, 22, 23] studied the global well-posedness and optimal time decay rate for the MHD system with the magnetic equilibrium state to be zero. Their studies get the optimal time decay rate for the derivatives of the solution which generalize the optimal time decay rate results for Navier-Stokes equations to the more complex MHD equations.

R. Danchin did a seminal work in [24] in 2000. By scaling analysis R. Danchin gave the definitions about critical space for compressible Navier-Stokes equations, then under this new framework he constructed a global solution when the initial data to be small in the critical Besov space. After this important work, there are a lot of further studies [4, 5, 25]. Especially, C. Wang, W. Wang and Z. Zhang [6] construct a global solution for isentropic Navier-Stokes system which allow highly oscillating initial density and velocity in 2014. In [26, 27], D. Bian, B. Yuan and B. Guo, firstly, constructed the local solution for compressible MHD equations in critical Besov space framework. In [19], C. Hao analyzed the linearized hyperbolic-parabolic system related to MHD equations carefully, then he constructed the global solution in critical Besov space framework. Recently, there are some important works for incompressible MHD equations [10, 11, 12]. In these excellent papers, F. Lin, L. Xu and P. Zhang constructed global solutions for incompressible MHD equations without magnetic diffusion by cutely using the dissipative structure of the system and the anisotropic Besov space technique. Then, X. Hu in paper [13] constructed the global solutions for compressible MHD equations without magnetic diffusion. Due to the structure is very different for incompressible and compressible MHD equations, X. Hu's work is very different to F. Lin, L. Xu and P. Zhang's work and X. Hu used lots of structure information about the compressible MHD equations.

In this manuscript, we plan to construct global solution for MHD system with highly oscillating initial velocity and density. For isentropic Navier-Stokes system, global solutions with highly oscillating initial velocity can be obtained by a careful analysis about Green's matrix for the linear system [4, 5]. In paper [33], the authors studied the global well-posedness for compressible viscoelastic fluids in L^p based critical Besov space by decomposing the whole system into three small systems with each one similar to Navier-Stokes case. However, as mentioned by A. Suen and D. Hoff in their paper [14], the presence of magnetic field effects results in a different and more intricate coupling of rates of partial regularization in the initial layer, so we need more careful analysis. we firstly transform compressible MHD system written in Euler coordinate into Lagrange coordinate in critical Besov space which is not appeared before. Then based on Lagrange coordinate form, we construct the local solution. Comparing to the known local results [26, 27], we improve the uniqueness range for p from $[2, N)$ to $[2, 2N)$ where N stands for dimension and we also give a lower bound for the maximal existence time which is important for our global result. Based on the local well-posedness, we obtain a global solution by using effective viscous flux and Hoff's energy estimates to explore the structure of

the system. For more explicit statements, please see Theorem 2.6 and Theorem 2.8 in next section.

The other parts of this paper are organized as follows. In section 2, we give a brief introduction about Besov space and the two main results of this paper. In Section 3, we prove the local well-posedness in critical Besov space framework by using Lagrange coordinate methods and prove that the local solution can propagate the smoothness of the initial data. In Section 4, we use Hoff's energy methods to get a uniform estimates about the solution. In Section 5, we prove a blow up criterion and then combine the results from Section 3 to complete the proof of Theorem 2.8. At last, for the reader's convenience, we collect a lot of useful Lemmas in the Appendix.

2. Main Results and Some Preliminaries

In this section, we will introduce some notations and give the three main results of this paper. Firstly, let me give some basic knowledge about Besov space, which can be found in [1]. The homogeneous Littlewood-Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\phi \in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C} := \{\xi \in \mathbb{R}^N, 3/4 \leq |\xi| \leq 8/3\}$ such that

$$\sum_{q \in \mathbb{Z}} \phi(2^{-q}\xi) = 1 \quad \text{if} \quad \xi \neq 0.$$

Denote $h = \mathcal{F}^{-1}\phi$, we then define the dyadic blocks as follows

$$\Delta_q u := \phi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^q y)u(x-y)\, dy, \quad \text{and} \quad S_q u = \sum_{k \leq q-1} \Delta_k u.$$

The formal decomposition

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u$$

is called homogeneous Littlewood-Paley decomposition. The above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of ϕ, we have

$$\Delta_k \Delta_q u = 0 \quad \text{if} \quad |k-q| \geq 2,$$
$$\Delta_k(S_{q-1}\Delta_q u) = 0 \quad \text{if} \quad |k-q| \geq 5.$$

Let us now introduce the homogeneous Besov space.

Definition 2.1. We denote by $\mathcal{S}_h'$ the space of tempered distributions u such that

$$\lim_{j \to -\infty} S_j u = 0 \quad \text{in} \quad \mathcal{S}'.$$

Definition 2.2. Let s be a real number and (p, r) be in $[1, \infty]^2$. The homogeneous Besov space $\dot{B}_{p,r}^s$ consists of distributions u in $\mathcal{S}_h'$ such that

$$\|u\|_{\dot{B}_{p,r}^s} := \left(\sum_{j \in \mathbb{Z}} 2^{rjs} \|\Delta_j u\|_{L^p}^r \right)^{1/r} < +\infty.$$

From now on, the notation $\dot{B}_p^s$ will stand for $\dot{B}_{p,1}^s$ and the notation $\dot{B}^s$ will stand for $\dot{B}_{2,1}^s$.

The study of non stationary PDE's usually requires spaces of type $L_T^r(X) := L^r(0, T; X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov spaces, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. However, in doing so, we obtain bounds in spaces which are not of type $L^r(0, T; \dot{B}_p^s)$. This approach was initiated in [2] and naturally leads to the following definitions:

Definition 2.3. Let $(r, p) \in [1, +\infty]^2$, $T \in (0, +\infty]$ and $s \in \mathbb{R}$. We set

$$\|u\|_{\tilde{L}_T^r(\dot{B}_p^s)} := \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q u(t)\|_{L^p}^r \, dt \right)^{1/r}$$

and

$$\tilde{L}_T^r(\dot{B}_p^s) := \left\{ u \in L_T^r(\dot{B}_p^s), \|u\|_{\tilde{L}_T^r(\dot{B}_p^s)} < +\infty \right\}.$$

Owing to Minkowski inequality, we have $\tilde{L}_T^r(\dot{B}_p^s) \hookrightarrow L_T^r(\dot{B}_p^s)$. That embedding is strict in general if $r > 1$. We will denote by $\tilde{C}_T(\dot{B}_p^s)$ the set of function u belonging to $\tilde{L}_T^\infty(\dot{B}_p^s) \cap C([0, T]; \dot{B}_p^s)$.

We will often use the following interpolation inequality:

$$\|u\|_{\tilde{L}_T^r(\dot{B}_p^s)} \leq \|u\|_{\tilde{L}_T^{r_1}(\dot{B}_p^{s_1})}^\theta \|u\|_{\tilde{L}_T^{r_2}(\dot{B}_p^{s_2})}^{1-\theta},$$

with

$$\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} \quad \text{and} \quad s = \theta s_1 + (1-\theta) s_2,$$

and the following embeddings

$$\tilde{L}_T^r(\dot{B}_p^{n/p}) \hookrightarrow L_T^r(\mathcal{C}_0) \quad \text{and} \quad \tilde{C}_T(\dot{B}_p^{n/p}) \hookrightarrow C([0, T] \times \mathbb{R}^N).$$

Another important space is the bybrid Besov space, we give the definitions and collect some properties.

Definition 2.4. let $s, t \in \mathbb{R}$. We set

$$\|u\|_{B_{q,p}^{s,t}} := \sum_{k \leq R_0} 2^{ks} \|\Delta_k u\|_{L^q} + \sum_{k > R_0} 2^{kt} \|\Delta_k u\|_{L^p}.$$

and

$$B_{q,p}^{s,t}(\mathbb{R}^N) := \left\{ u \in \mathcal{S}_h'(\mathbb{R}^N), \|u\|_{B_{q,p}^{s,t}} < +\infty \right\},$$

where R_0 is a fixed constant.

Lemma 2.5. 1) We have $B_{2,2}^{s,s} = \dot{B}^s$.
2) If $s \leq t$ then $B_{p,p}^{s,t} = B_{p,p}^s \cap B_{p,p}^t$. Otherwise, $B_{p,p}^{s,t} = B_{p,p}^s + B_{p,p}^t$.
3) The space $B_{p,p}^{0,s}$ coincide with the usual inhomogeneous Besov space.
4) If $s_1 \leq s_2$ and $t_1 \geq t_2$ then $B_{p,p}^{s_1,t_1} \hookrightarrow B_{p,p}^{s_2,t_2}$.
5) Interpolation: For $s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R}$ and $\theta \in [0, 1]$, we have

$$\|f\|_{B_{2,p}^{\theta s_1 + (1-\theta) s_2, \theta \sigma_1 + (1-\theta) \sigma_2}} \leq \|f\|_{B_{2,p}^{s_1, \sigma_1}}^\theta \|f\|_{B_{2,p}^{s_2, \sigma_2}}^{1-\theta}.$$

From now on, the notation $B_p^{s,t}$ will stand for $B_{p,p}^{s,t}$ and the notation $B^{s,t}$ will stand for $B_{2,2}^{s,t}$.

Throughout the paper, we shall use some paradifferential calculus. It is a nice way to define a generalized product between distributions which is continuous in functional spaces where the usual product does not make sense. The paraproduct between u and v is defined by

$$T_u v := \sum_{q \in \mathbb{Z}} S_{q-1} u \Delta_q v.$$

We thus have the following formal decomposition:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$R(u,v) := \sum_{q \in \mathbb{Z}} \Delta_q u \tilde{\Delta}_q v, \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

We will sometimes use the notation $T'_u v := T_u v + R(u,v)$.

For more information about Besov space and hybrid Besov space, we give reference [1, 3, 4, 5]. Throughout this paper, C stands for a "harmless" constant, and we sometimes use the notation $A \lesssim B$ as an equivalent of $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

With these preparations, we can state our two main results.

Theorem 2.6. *Let $\bar{\rho} > 0$ and $c_0 > 0$. Assume that the initial data (ρ_0, u_0, B_0) satisfies*

$$\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{N/p}, \quad c_0 \leq \rho_0 \leq c_0^{-1},$$
$$u_0, B_0 \in \dot{B}_{p,1}^{N/p-1}.$$

Then there exists a positive time $T > 0$ such that if $p \in [2, 2N)$, the system has a unique solution $(\rho - \bar{\rho}, u, B)$ satisfies

$$\rho - \bar{\rho} \in \tilde{C}([0,T]; \dot{B}_{p,1}^{N/p}), \quad \frac{1}{2}c_0 \leq \rho \leq 2c_0^{-1},$$
$$u, B \in \tilde{C}([0,T]; \dot{B}_{p,1}^{N/p-1}) \cap L^1(0,T; \dot{B}_{p,1}^{N/p+1}).$$

In addition, we can take the maximal existence time T as follows

$$T = \frac{\bar{c}}{(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^4},$$

where $\bar{c}$ is a small enough positive constant.

Remark 2.7. The above results are necessary for our following global well-posedness results, however, Theorem 2.6 has its own interest. As C. Noboru and R. Danchin [29] improved the uniqueness condition for Navier-Stokes system, we improve the uniqueness condition from $p \in [2, N)$ [26, 27] to $p \in [2, 2N)$ by using Lagrange coordinate technique. In our case, we need to get the magnetic field equation in Lagrange coordinate and estimate more complex coupling terms. In addition, we get a lower bound for the maximal existence time which is necessary for our global existence theorem.

Theorem 2.8. *Suppose the equilibrium state $\bar{\rho} > 0$, $\bar{B} = 0$ and dimension $N = 3$. Let c_0 to be a small constant,*

$$P_+ = \sup_{c_0/4 \le \rho \le 4c_0^{-1}} |P^{(k)}(\rho)| \quad and \quad P_- = \inf_{c_0/4 \le \rho \le 4c_0^{-1}} |P'(\rho)|.$$

Assume that the initial data (ρ_0, u_0) satisfies

$$\rho_0 - \bar{\rho} \in H^s \cap \dot{B}_{p,1}^{3/p}, \quad c_0 \le \rho_0 \le c_0^{-1},$$

$$u_0, B_0 \in H^{s-1} \cap \dot{B}_{p,1}^{3/p-1},$$

for some $p \in (3, 6)$ and $s \ge 3$. There exist five constants c_1, c_2, c_3 such that

$$\|\rho_0 - \bar{\rho}\|_{L^2} \le c_1, \quad \|u_0\|_{\dot{B}_{p,1}^{3/p-1}} \le \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^5}, \quad \|B_0\|_{\dot{B}_{p,1}^{3/p-1}} \le \frac{c_3}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^2},$$

$$\|u_0\|_{\dot{H}^{-\delta}} \le \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^5 (1 + \|\rho_0 - \bar{\rho}\|_{H^2}^8 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})},$$

$$\|B_0\|_{\dot{H}^{-\delta}} \le \frac{c_3}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^3 (1 + \|\rho_0 - \bar{\rho}\|_{H^2}^8 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})},$$

for some $\delta \in \left(1 - \frac{3}{p}, \frac{3}{p}\right)$, then there exist a unique global solution (ρ, u, B) satisfying

$$\rho \ge \frac{c_0}{4}, \quad \rho - \bar{\rho} \in C([0, \infty); H^s),$$

$$u, B \in C([0, \infty); H^{s-1}) \cap L^2(0, T; H^s),$$

for any $T > 0$.

The proof of the above theorem used the advantage of harmonic analysis and the structure variable called effective viscous flux. The magnetic field and velocity field coupled with each other in the compressible MHD system so we need to estimate many coupled terms. Based on the Theorem 2.6, we used energy methods in [14] which can incorporate the structure conditions of our system. However, our regularity conditions are better than the conditions in [14], so we need not add additional conditions for viscosity coefficient to extend the local solution to a global one.

Remark 2.9. Isentropic Navier-Stokes system is well-posedness when the initial velocity field have large oscillation which is proved in [4, 5]. Until now, there are no such results for compressible MHD systems for the coupling between velocity and magnetic field make MHD system are more complex than Navier-Stokes equations. The above theorem allows the initial velocity field and magnetic field have large oscillation and allows the initial density to have large oscillation on the set of small measure. Comparing to previous results for MHD system, the theorem above highly reduce the requirements about the small condition on the initial data.

3. LOCAL EXISTENCE AND PROPAGATION OF REGULARITY

In this section, we will prove Theorem 2.6 and prove that the obtained solution can propagate the smoothness of the initial data. Without loss of generality, we can assume $\bar{\rho} = 1$ from this section to the last section.

Let X_u be the flow associated to the vector field u, that is the solution to

$$(3.1) \qquad X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau, y))\, d\tau.$$

Denoting

$$\tilde{\rho}(t,y) = \rho(t, X_u(t,y)), \quad \tilde{u}(t,y) = u(t, X_u(t,y)), \quad \tilde{B}(t,y) = B(t, X_u(t,y)).$$

Then we reformulate the MHD system in the Lagrangian coordinate to obtain

$$(3.2) \qquad \begin{cases} \partial_t(J_u\tilde{\rho}) = 0, \\ J_u\rho_0\partial_t\tilde{u} - \mathrm{div}\left(\mathrm{adj}(DX_u)\left(2\mu D_{A_u}\tilde{u} + \lambda\mathrm{div}_{A_u}\tilde{u}\,\mathrm{Id} + P(\tilde{\rho})\mathrm{Id}\right)\right) \\ \qquad\qquad = \mathrm{div}\left(\mathrm{adj}(DX_u)\tilde{B}\tilde{B}^T\right) - \frac{1}{2}\mathrm{div}\left(\mathrm{adj}(DX_u)|\tilde{B}|^2\right), \\ J_u\partial_t\tilde{B} - \nu\mathrm{div}\left(\mathrm{adj}(DX_u)A_u^T\nabla\tilde{B}\right) \\ \qquad\qquad = \mathrm{div}\left(\mathrm{adj}(DX_u)\tilde{B}\tilde{u}^T\right) - \mathrm{div}\left(\mathrm{adj}(DX_u)\tilde{u}\tilde{B}\right), \\ \mathrm{div}(A_u\tilde{B}) = 0, \end{cases}$$

where $A_u = (D_y X_u)^{-1}$. During the transformation between Euler coordinates and Lagrangian coordinate, we used (6.1).

Next, we introduce u_L the "free solution" to

$$(3.3) \qquad \begin{cases} \partial_t u_L - \mu\Delta u_L - (\lambda + \mu)\nabla\mathrm{div}u_L = 0, \\ u_L|_{t=0} = u_0, \end{cases}$$

and B_L the "free solution" to

$$(3.4) \qquad \begin{cases} \partial_t B_L - \nu\Delta B_L = 0, \\ \mathrm{div}B_L = 0, \\ B_L|_{t=0} = B_0. \end{cases}$$

Denoting

$$(3.5) \qquad L_{\rho_0}(\tilde{u}) = \partial_t\tilde{u} - \frac{1}{\rho_0}\mathrm{div}\left(2\mu\mathcal{D}(\tilde{u}) + \lambda\mathrm{div}\tilde{u}\,\mathrm{Id}\right),$$

and

$$(3.6) \qquad L_\nu(\tilde{B}) = \partial_t\tilde{B} - \nu\Delta\tilde{B}.$$

We can rewrite the second and third equations in system (3.2) as follows

$$(3.7) \qquad \begin{cases} L_{\rho_0}(\tilde{u}) = I_1(\tilde{u}, \tilde{u}) + \rho_0^{-1}\mathrm{div}(I_2(\tilde{u}, \tilde{u}) + I_3(\tilde{u}, \tilde{u}) + I_4(\tilde{u})) \\ \qquad\qquad\qquad + \rho_0^{-1}\mathrm{div}(I_5(\tilde{u}, \tilde{B}) + I_6(\tilde{u}, \tilde{B})), \\ L_\nu(\tilde{B}) = I_7(\tilde{u}, \tilde{B}) + \mathrm{div}(I_8(\tilde{u}, \tilde{B}) + I_9(\tilde{u}, \tilde{B}) \\ \qquad\qquad\qquad + I_{10}(\tilde{u}, \tilde{B}, \tilde{u}) + I_{11}(\tilde{u}, \tilde{B}, \tilde{u})), \end{cases}$$

where

$$I_1(v,w) = (1 - J_v)\partial_t w,$$

$$I_2(v,w) = (\mathrm{adj}(DX_v) - Id)(\mu Dw \cdot A_v + \mu A_v^T \nabla w + \lambda A_v^T : \nabla w),$$

$$I_3(v,w) = \mu Dw \cdot (A_v - Id) + \mu(A_v^T - Id)\nabla w + \lambda(A_v^T - Id) : \nabla w,$$

$$I_4(v) = \mathrm{adj}(DX_v)P(J_v^{-1}\rho_0)\,Id, \quad I_5(v,w) = \mathrm{adj}(DX_v)ww^T,$$

$$I_6(v,w) = \frac{1}{2}\mathrm{adj}(DX_v)|w|^2, \quad I_7(v,w) = (1 - J_v)\partial_t w,$$

$$I_8(v,w) = \nu(\mathrm{adj}(DX_v) - Id)A_v^T \nabla w, \quad I_9(v,w) = \nu(A_v^T - Id)\nabla w,$$

$$I_{10}(v,w,k) = \mathrm{adj}(\nabla X_v)wk^T, \quad I_{11}(v,w,k) = \mathrm{adj}(DX_v)wk.$$

In order to prove the local well-posedness, we define a map $\mathbf{\Phi} : (v,b) \mapsto (\tilde{u}, \tilde{B})$ where $(\tilde{u}, \tilde{B})$ satisfies (3.7) with the $\tilde{u}, \tilde{B}$ on the right hand side changed to v, b separately. Denoting $\hat{u} = \tilde{u} - u_L$ and $\hat{B} = \tilde{B} - B_L$, then we know that $\hat{u}$ and $\hat{B}$ satisfies

$$(3.8) \quad \begin{cases} L_{\rho_0}(\hat{u}) = (L_1 - L_{\rho_0})u_L + I_1(v,v) \\ \qquad\qquad + \rho_0^{-1}\mathrm{div}(I_2(v,v) + I_3(v,v) + I_4(v) + I_5(v,b) + I_6(v,b)), \\ L_\nu(\hat{B}) = I_7(v,b) + \mathrm{div}(I_8(v,b) + I_9(v,b) + I_{10}(v,b,v) + I_{11}(v,b,v)), \\ \mathrm{div}(\hat{B}) = \mathrm{div}((Id - A_v)b), \\ (\hat{u}, \hat{B})|_{t=0} = (0,0). \end{cases}$$

In order to give a clear statement, we need to define

$$E_p(T) = \left\{ v \in \tilde{L}([0,T]; \dot{B}_{p,1}^{N/p-1}) : \partial_t v, \nabla^2 v \in \tilde{L}^1(0,T; \dot{B}_{p,1}^{N/p-1}) \right\},$$

$$\bar{B}_{E_p(T)}(u_L, B_L; R) = \left\{ v,b : \|v - u_L\|_{E_p(T)} + \|b - B_L\|_{E_p(T)} \leq R \right\}.$$

Suppose that $T, R \leq 1$, and T, R small enough such that

$$(3.9) \qquad\qquad \int_0^T \|\nabla v\|_{\dot{B}_{p,1}^{N/p}}\, dt \leq c < 1.$$

Step 1 : Stability of $\bar{\mathbf{B}}_{\mathbf{E_p(T)}}(\mathbf{u_L}, \mathbf{B_L}; \mathbf{R})$ for small enough R and T. Using Lemma 6.1, we could get

$$\|\hat{u}\|_{E_p(T)} \leq Ce^{C_{\rho_0,m}T}\Big(\|I_1(v,v)\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} + \|(L_1 - L_{\rho_0})u_L\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} \Big)$$

$$+ (1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})\Big(\|I_2(v,v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} + \|I_3(v,v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} $$

$$+ \|I_4(v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} + \|I_5(v,b)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} + \|I_6(v,b)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \Big).$$

Using Lemma 6.2, we could get

$$\|\hat{B}\|_{E_p(T)} \leq C\Big(\|I_7(v,b)\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} + \|I_8(v,b)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} + \|I_9(v,b)\|_{L_T^1(\dot{B}_{p,1}^{N/p})}$$

$$+ \|I_{10}(v,b,v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} + \|I_{11}(v,b,v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \Big).$$

There are many terms can be estimated similar to Navier-Stokes system as follows

$$\|I_1(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} \leq C\|v\|_{L_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t w\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})},$$

$$\|(L_1 - L_{\rho_0})u_L\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} \leq C\|a_0\|_{\dot{B}_{p,1}^{N/p}}(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})\|u_L\|_{L_T^1(\dot{B}_{p,1}^{N/p+1})},$$

$$\|I_i(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})\|v\|_{L_T^1(\dot{B}_{p,1}^{N/p})}\|w\|_{L_T^1(\dot{B}_{p,1}^{N/p})}, \quad (i = 2,3),$$

$$\|I_4(v)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq CT(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})(1 + \|v\|_{L_T^1(\dot{B}_{p,1}^{N/p})}).$$

Here, we only give the estimates for the terms different to Navier-Stokes systems.

$$\|I_5(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C\left(\|\mathrm{adj}(DX_v) - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})} + 1\right)\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}^2$$
$$\leq C(1 + \|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}^2,$$

$$\|I_6(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C(1 + \|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}^2,$$

$$\|I_7(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p-1})} \leq C\|1 - J_v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})}\|\partial_t w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}$$
$$\leq C\|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})},$$

$$\|I_8(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C\|\mathrm{adj}(DX_v) - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})}\|A_v^T - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})}\|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$+ C\|\mathrm{adj}(DX_v) - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})}\|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})}$$
$$\leq C\|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})},$$

$$\|I_9(v,w)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C\|A_v^T - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})}\|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$\leq C\|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})},$$

$$\|I_{10}(v,w,k)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C(1 + \|\mathrm{adj}(\nabla X_v) - Id\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{N/p})})\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|k\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}$$
$$\leq C(1 + \|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})})\|k\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})},$$

$$\|I_{11}(v,w,k)\|_{L_T^1(\dot{B}_{p,1}^{N/p})} \leq C(1 + \|v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})})\|k\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|w\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}.$$

Combining the estimates about I_1 to I_{11}, we could finally get

$$\|\hat{u}\|_{E_p(T)} \leq Ce^{C_{\rho_0,m}T}(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^2\Big\{T + \|a_0\|_{\dot{B}_{p,1}^{N/p}}\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$+ (\|\hat{v}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})(\|\partial_t\hat{v}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} + \|\partial_t u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})})$$
$$+ (\|\hat{v}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})^2$$
$$+ (\|\hat{b}\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} + \|b_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})})^2\Big\},$$

and

$$\|\hat{B}\|_{E_p(T)} \leq C\Big((\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|\hat{v}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})(\|\partial_t B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}$$
$$+ \|\partial_t\hat{b}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} + \|\hat{b}\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})$$
$$+ (\|\hat{v}\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} + \|v_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})})(\|\hat{b}\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} + \|B_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})})\Big),$$

where $\hat{v} = v - u_L$, $\hat{b} = b - B_L$.

We could reduce the above two inequalities further to get

$$
\begin{aligned}
\|\hat{u}\|_{E_p(T)} \leq & C e^{C_{\rho_0,m}T}(1+\|a_0\|_{\dot{B}_{p,1}^{N/p}})^2\Big\{T+\|a_0\|_{\dot{B}_{p,1}^{N/p}}\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} \\
& + (R+\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})})(R+\|\partial_t u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \\
& + \|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}) + (R+\|B_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})})^2\Big\},
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\|\hat{B}\|_{E_p(T)} \leq & C(R+\|u_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}+\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})})(R+\|\partial_t B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \\
& + \|B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}+\|B_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}).
\end{aligned}
\tag{3.11}
$$

Here, if we assume

$$
\begin{aligned}
& C_{\rho_0,m}T \leq \log 2, \quad T \leq R^2, \\
& \|a_0\|_{\dot{B}_{p,1}^{N/p}}\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \leq R^2, \\
& \|\partial_t u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}+\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}+\|u_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} \leq R, \\
& \|\partial_t B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}+\|B_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}+\|B_L\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} \leq R, \\
& (1+\|a_0\|_{\dot{B}_{p,1}^{N/p}})^2 R \leq \eta < \frac{1}{20C},
\end{aligned}
\tag{3.12}
$$

then we know that

$$
\|\hat{u}\|_{E_p(T)} \leq R, \quad \|\hat{B}\|_{E_p(T)} \leq R.
$$

Hence, $\boldsymbol{\Phi}$ is a self map on the ball $\bar{B}_{E_p(T)}(u_L, B_L; R)$.

Step 2 : Contraction estimates. Taking $(v^1, b^1) \in \bar{B}_{E_p(T)}(u_L, B_L; R)$ and $(v^2, b^2) \in \bar{B}_{E_p(T)}(u_L, B_L; R)$, we define $(\tilde{u}^1, \tilde{B}^1) = \boldsymbol{\Phi}(v^1, b^1)$ and $(\tilde{u}^2, \tilde{B}^2) = \boldsymbol{\Phi}(v^1, b^1)$. Denote $\delta\tilde{u} = \tilde{u}^2 - \tilde{u}^1$, $\delta\tilde{B} = \tilde{B}^2 - \tilde{B}^1$, $\delta v = v^2 - v^1$ and $\delta b = b^2 - b^1$.

Through simple calculations, we have

$$
\begin{aligned}
L_{\rho_0}(\delta\tilde{u}) = & I_1(v^1, \delta v) + (J_{v^1} - J_{v^2})\partial_t v^2 \\
& + \rho_0^{-1}\Big\{(I_2(v^2, v^2) - I_2(v^1, v^1)) + (I_3(v^2, v^2) - I_3(v^1, v^1)) \\
& + (I_4(v^2) - I_4(v^1)) + (I_5(v^2, b^2) - I_5(v^1, b^1)) \\
& + (I_6(v^2, b^2) - I_6(v^1, b^1))\Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
L_\nu(\delta\tilde{B}) = & (I_7(v^2, b^2) - I_7(v^1, b^1)) + \operatorname{div}\Big\{(I_8(v^2, b^2) - I_8(v^1, b^1)) \\
& + (I_9(v^2, b^2) - I_9(v^1, b^1)) + (I_{10}(v^2, b^2, v^2) - I_{10}(v^1, b^1, v^1)) \\
& + (I_{11}(v^2, b^2, v^2) - I_{11}(v^1, b^1, v^1))\Big\}.
\end{aligned}
$$

Under the condition $C_{\rho_0,m}T \le \log 2$, using Lemma 6.1 and Lemma 6.2, we get

$$
\begin{aligned}
\|\delta\tilde{u}\|_{E_p(T)} \le &C(1+\|a_0\|_{\dot{B}_{p,1}^{N/p}})\Big(\|I_1(v^1,\delta v)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \\
&+\|(J_{v^1}-J_{v^2})\partial_t v^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}+\|I_2(v^2,v^2)-I_2(v^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_3(v^2,v^2)-I_3(v^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})}+\|I_4(v^2)-I_4(v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_5(v^2,b^2)-I_5(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_6(v^2,b^2)-I_6(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\delta\tilde{B}\|_{E_p(T)} \le &C\Big(\|I_7(v^2,b^2)-I_7(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \\
&+\|I_8(v^2,b^2)-I_8(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_9(v^2,b^2)-I_9(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_{10}(v^2,b^2,v^2)-I_{10}(v^1,b^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \\
&+\|I_{11}(v^2,b^2,v^2)-I_{11}(v^1,b^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})}\Big).
\end{aligned}
$$

There are many terms appeared in the Navier-Stokes system, we only list the estimates

$$\|I_1(v^1,\delta v)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \le C\|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t \delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})},$$

$$\|(J_{v^1}-J_{v^2})\partial_t v^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} \le C\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t v^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})},$$

$$\|I_2(v^2,v^2)-I_2(v^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \le C\|(v^1,v^2)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})},$$

$$\|I_3(v^2,v^2)-I_3(v^1,v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \le C\|(v^1,v^2)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})},$$

$$\|I_4(v^2)-I_4(v^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \le C(1+\|a_0\|_{\dot{B}_{p,1}^{N/p}})T\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}.$$

Now, we analyze the different terms carefully. Since

$$
\begin{aligned}
I_5(v^2,b^2)-I_5(v^1,b^1) =&(\mathrm{adj}(DX_{v^2})-\mathrm{adj}(DX_{v^1}))b^2(b^2)^T \\
&+\mathrm{adj}(DX_{v^1})(b^2-b^1)(b^2)^T \\
&+\mathrm{adj}(DX_{v^1})b^1(b^2-b^1)^T,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|I_5(v^2,b^2)-I_5(v^1,b^1)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} \le &C\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|b^2\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}^2 \\
&+C(1+\|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})})\|\delta b\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|(b^1,b^2)\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}.
\end{aligned}
$$

I_6 can be estimated same as I_5. Since

$$I_7(v^2,b^2)-I_7(v^1,b^1) = (J_{v^1}-J_{v^2})\partial_t b^2 + (1-J_{v^1})(\partial_t b^2-\partial_t b^1),$$

we know that

$$\begin{aligned}
\|I_7(v^2,b^2) - I_7(v^1,b^1)\|_{\tilde{L}^1_T(\dot{B}^{N/p-1}_{p,1})} \leq &C\|J_{v^2} - J_{v^1}\|_{\tilde{L}^\infty_T(\dot{B}^{N/p}_{p,1})}\|\partial_t b^2\|_{\tilde{L}^1_T(\dot{B}^{N/p-1}_{p,1})} \\
&+ C\|J_{v^1} - 1\|_{\tilde{L}^\infty_T(\dot{B}^{N/p}_{p,1})}\|\partial_t \delta b\|_{\tilde{L}^1_T(\dot{B}^{N/p-1}_{p,1})} \\
\leq &C\|\partial_t b^2\|_{\tilde{L}^1_T(\dot{B}^{N/p-1}_{p,1})}\|\delta v\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})} \\
&+ C\|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})}\|\partial_t \delta b\|_{\tilde{L}^1_T(\dot{B}^{N/p-1}_{p,1})}.
\end{aligned}$$

After some calculations, we have

$$\begin{aligned}
I_8(v^2,b^2) - I_8(v^1,b^1) =&\nu(\mathrm{adj}(DX_{v^2}) - \mathrm{adj}(DX_{v^1}))A^T_{v^2}\nabla b^2 \\
&+ \nu(\mathrm{adj}(DX_{v^1}) - Id)(A^T_{v^2} - A^T_{v^1})\nabla b^2 \\
&+ \nu(\mathrm{adj}(DX_{v^1}) - Id)A^T_{v^1}\nabla(b^2 - b^1),
\end{aligned}$$

so we can get

$$\begin{aligned}
&\|I_8(v^2,b^2) - I_8(v^1,b^1)\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})} \\
\leq &C\|\delta v\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}(1 + \|v^2\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})})\|b^2\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})} \\
&+ C\|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}\|\delta v\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}\|b^2\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})} \\
&+ C\|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}(1 + \|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})})\|\delta b\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}.
\end{aligned}$$

Due to

$$I_9(v^2,b^2) - I_9(v^1,b^1) = \nu(A^T_{v^2} - A^T_{v^1})\nabla b^2 + \nu(A^T_{v^1} - Id)\nabla\delta b,$$

we have

$$\begin{aligned}
\|I_9(v^2,b^2) - I_9(v^1,b^1)\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})} \leq &C\|\delta v\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}\|b^2\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})} \\
&+ C\|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}\|\delta b\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}.
\end{aligned}$$

Since

$$\begin{aligned}
I_{10}(v^2,b^2,v^2) - I_{10}(v^1,b^1,v^1) =&(\mathrm{adj}(\nabla X_{v^2}) - \mathrm{adj}(\nabla X_{v^1}))b^2(v^2)^T \\
&+ \mathrm{adj}(\nabla X_{v^1})(b^2 - b^1)(v^2)^T \\
&+ \mathrm{adj}(\nabla X_{v^1})b^1(v^2 - v^1)^T,
\end{aligned}$$

we easily have

$$\begin{aligned}
&\|I_{10}(v^2,b^2,v^2) - I_{10}(v^1,b^1,v^1)\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})} \\
\leq &C\|\delta v\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})}\|b^2\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})}\|v^2\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})} \\
&+ C(1 + \|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})})\|\delta b\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})}\|v^2\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})} \\
&+ C(1 + \|v^1\|_{\tilde{L}^1_T(\dot{B}^{N/p+1}_{p,1})})\|b^1\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})}\|\delta v\|_{\tilde{L}^2_T(\dot{B}^{N/p}_{p,1})}.
\end{aligned}$$

The term $\|I_{11}(v^2,b^2,v^2) - I_{11}(v^1,b^1,v^1)\|_{\tilde{L}^1_T(\dot{B}^{N/p}_{p,1})}$ can be estimated similar to the above terms, so we not write the details. Combining all the above estimates, we

finally arrive at

$$\|\delta\tilde{u}\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^2 \Big\{ (T + \|(v^1, v^2)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$+ \|\partial_t v^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} + \|b^2\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}^2)\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})}$$
$$+ \|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t \delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} + \|(b^1, b^2)\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|\delta b\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} \Big\},$$

and

$$\|\delta\tilde{B}\|_{E_p(T)} \leq C\Big\{ (\|\partial_t b^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})} + \|b^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|b^2\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$+ \|v^2\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|b^2\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})})\|\delta v\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}$$
$$+ \|b^1\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|\delta v\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} + \|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\partial_t \delta b\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p-1})}$$
$$+ \|v^1\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})}\|\delta b\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p+1})} + \|v^2\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})}\|\delta b\|_{\tilde{L}_T^2(\dot{B}_{p,1}^{N/p})} \Big\}.$$

Through some simple calculations, using conditions (3.12) with may be lager constant C, we have

$$\|\delta\tilde{u}\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^2 R(\|\delta v\|_{E_p(T)} + \|\delta b\|_{E_p(T)}),$$

and

$$\|\delta\tilde{B}\|_{E_p(T)} \leq CR(\|\delta v\|_{E_p(T)} + \|\delta b\|_{E_p(T)}).$$

Hence, we get

$$\|(\delta\tilde{u}, \delta\tilde{B})\|_{E_p(T)} \leq C(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^2 R\|(\delta v, \delta b)\|_{E_p(T)}.$$

Due to conditions (3.12), we finally get

$$(3.13) \qquad \|(\delta\tilde{u}, \delta\tilde{B})\|_{E_p(T)} \leq \frac{1}{2}\|(\delta v, \delta b)\|_{E_p(T)}.$$

Now, by contraction mapping theorem, we know that $\boldsymbol{\Phi}$ admits a unique fixed point in $\bar{B}_{E_p(T)}(u_L, B_L; R)$.

Step 3 : Regularity of the density. Denoting $a = \rho - 1$, and we already know that $\rho = J_u^{-1}\rho_0$. So we have

$$a = (J_u^{-1} - 1)a_0 + a_0 + (J_u^{-1} - 1).$$

Due to $u \in L^1(0, T; \dot{B}_{p,1}^{N/p+1})$, we know that $J_u^{-1}(t) - 1 \in C([0, T]; \dot{B}_{p,1}^{N/p})$. So we know that a belongs to $C([0, T]; \dot{B}_{p,1}^{N/p})$. If

$$(3.14) \qquad R \leq \frac{c_0}{4(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})},$$

we have

$$\rho(t) \geq (1 + a_0) - (1 + \|a_0\|_{L^\infty})(\|u_L\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{N/p})} + R)$$
$$\geq (1 + a_0) - 2(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})R$$
$$> 0,$$

where c_0 defined as in Theorem 2.8 and $t \in [0, T]$.

Step 4 : Uniqueness and continuity of the flow map. Consider two couples of initial data (ρ_0^1, u_0^1, B_0^1) and (ρ_0^2, u_0^2, B_0^2). Define $\delta\rho = \rho^2 - \rho^1$, $\delta u = u^2 - u^1$

and $\delta B = B^2 - B^1$, we can easily find the equations for δu and δB. Using similar methods as Navier-Stokes system and estimate the terms containing B as in Step 3, we can finish this part. For there are no new gradients, we omit the details.

At this stage, by same reasons as in [29], we can get the local well-posedness results for our MHD system.

Step 5 : Lower bound of the existence time. Taking m to be a large constant fixed as before and C to be a large enough constant, let

$$(3.15) \qquad \eta \le \frac{1}{C}, \quad C_{\rho_0,m}T \le \log 2, \quad (1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^2 R \le \eta, \quad T \le R^2,$$
$$\|a_0\|_{\dot{B}_{p,1}^{N/p}}\|u_0\|_{\dot{B}_{p,1}^{N/p-1}} \le R^2, \quad \|u_0\|_{\dot{B}_{p,1}^{N/p-1}} \le R, \quad \|B_0\|_{\dot{B}_{p,1}^{N/p-1}} \le R,$$

then the conditions (3.12) and (3.14) are satisfied. From the above conditions, we know that

$$(3.16) \qquad T \le \frac{1}{C^2(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^4}, \quad T \le \frac{\log 2}{C_{\rho_0,m}}.$$

Next, we calculate $C_{\rho_0,m}$ carefully as follows

$$
\begin{aligned}
C_{\rho_0,m} &\le C \int_0^t \left\| S_m \nabla \left(\frac{1}{\rho_0} \right) \right\|_{\dot{B}_{p,1}^{N/p}}^2 d\tau \\
&\le C \int_0^t \| S_m \nabla \rho_0 \|_{\dot{B}_{p,1}^{N/p}}^2 d\tau \\
&\le C T 2^{2m} \|a_0\|_{\dot{B}_{p,1}^{N/p}}^2
\end{aligned}
$$
(3.17)

Combining (3.16) and (3.17), we know that we can take T as

$$T = \frac{\bar{c}}{(1 + \|a_0\|_{\dot{B}_{p,1}^{N/p}})^4},$$

where $\bar{c}$ is a small enough positive constant. At this point, we finished our proof of Theorem 2.6.

Next, let us go to the second part of this section to prove the solution in Theorem 2.6 can propagate the smoothness of the initial data.

Before our proof, we need to introduce some notations which can also be found in [6, 30]. Define a weight function $\{\omega_k(t)\}_{k \in \mathbb{Z}}$ as follows

$$\omega_k(t) = \sum_{\ell \ge k} 2^{k-\ell}(1 - e^{-c 2^{2j} t})^{1/2}, \quad k \in \mathbb{Z},$$

where c is a positive constant. We easily know that for any $k \in \mathbb{Z}$,

$$(3.18) \qquad \begin{aligned} &\omega_k(t) \le 2, \quad \omega_k(t) \sim \omega_{k'}(t) \quad \text{if } k \sim k', \\ &\omega_k(t) \le 2^{k-k'}\omega_{k'}(t) \quad \text{if } k \ge k', \quad \omega_k(t) \le 3\omega_{k'}(t) \quad \text{if } k \le k'. \end{aligned}$$

Now, we can introduce the following weighted Besov space.

Definition 3.1. Let $s \in \mathbb{R}$, $1 \le p, r \le +\infty$, $0 < T < +\infty$. The weighted Besov space $\dot{B}_{p,r}^s(\omega)$ is defined by

$$\dot{B}_{p,r}^s(\omega) = \left\{ f \in \mathcal{S}_h' \ : \ \|f\|_{\dot{B}_{p,r}^s(\omega)} < +\infty \right\},$$

where

$$\|f\|_{\dot{B}_{p,r}^s(\omega)} := \left\|\left(2^{ks}\omega_k(T)\|\Delta_k f\|_{L^p}\right)_k\right\|.$$

Obviously, $\dot{B}_{p,r}^s \subset \dot{B}_{p,r}^s(\omega)$ and

$$\|f\|_{\dot{B}_{p,r}^s(\omega)} \leq 2\|f\|_{\dot{B}_{p,r}^s}.$$

We also need to define the Chemin-Lerner type weighted Besov spaces as follows.

Definition 3.2. Let $s \in \mathbb{R}$, $1 \leq p,q,r \leq +\infty$, $0 < T \leq +\infty$. The weighted functional space $\tilde{L}_T^q(\dot{B}_{p,r}^s(\omega))$ is defined as the set of all the distributions f satisfying

$$\|f\|_{\tilde{L}_T^q(\dot{B}_{p,r}^s(\omega))} := \left\|\left(2^{ks}\omega_k(T)\|\Delta_k f(t)\|_{L^q(0,T;L^p)}\right)_k\right\|_{\ell^r}.$$

After this short introduction about weighted Besov space, we deduce some properties for the solution we obtained in the first part of this section. Obviously, we have

$$(3.19) \qquad\qquad \frac{c_0}{2} \leq \rho(t,x) \leq 2c_0^{-1}.$$

Since

$$\|\tilde{u}\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p-1})} + \|\tilde{u}\|_{\tilde{L}_T^1(\dot{B}_p^{N/p+1})} \leq R \leq \frac{c_3}{(1+\|a_0\|_{\dot{B}_p^{N/p}})^2},$$

and

$$\|u_L\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p-1})} + \|u_L\|_{\tilde{L}_T^1(\dot{B}_p^{N/p+1})} \leq \|u_0\|_{\dot{B}_p^{N/p-1}} \leq \frac{c_2}{(1+\|a_0\|_{\dot{B}_p^{N/p}})^5},$$

we know that

$$(3.20) \qquad \|u\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p-1})} + \|u\|_{\tilde{L}_T^1(\dot{B}_p^{N/p+1})} \leq \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{N/p}})^5}.$$

For the magnetic field, we have

$$(3.21) \quad \|B\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p-1})} + \|B\|_{\tilde{L}_T^1(\dot{B}_p^{N/p+1})} \leq R + \|B_0\|_{\dot{B}_p^{N/p}} \leq \frac{2c_3}{(1+\|a_0\|_{\dot{B}_p^{N/p}})^2}.$$

For the density, we know that

$$\|a\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p})} \leq \|Du\|_{\tilde{L}_T^1(\dot{B}_p^{N/p})}\|a_0\|_{\dot{B}_p^{N/p}} + \|a_0\|_{\dot{B}_p^{N/p}} + \|Du\|_{\tilde{L}_T^1(\dot{B}_p^{N/p})}.$$

Hence, by some simple calculations, we obtain

$$(3.22) \qquad \|a\|_{\tilde{L}_T^\infty(\dot{B}_p^{N/p})} \leq 2\|a_0\|_{\dot{B}_p^{N/p}} + \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{N/p}})^5}.$$

So combining (3.19), (3.20), (3.21) and (3.22), for dimension $N=3$, we know that the solution satisfies

$$\frac{c_0}{2} \leq \rho(t,x) \leq 2c_0^{-1},$$

$$\|a\|_{\tilde{L}_T^\infty(\dot{B}_p^{3/p})} \leq 2\|a_0\|_{\dot{B}_p^{3/p}} + \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^5},$$

$$(3.23)$$

$$\|u\|_{\tilde{L}_T^\infty(\dot{B}_p^{3/p-1})} + \|u\|_{\tilde{L}_T^1(\dot{B}_p^{3/p+1})} \leq \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^5},$$

$$\|B\|_{\tilde{L}_T^\infty(\dot{B}_p^{3/p-1})} + \|B\|_{\tilde{L}_T^1(\dot{B}_p^{3/p+1})} \leq \frac{2c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^2}.$$

At this stage, we estimate $\|a\|_{\tilde{L}_T^\infty(\dot{B}_p^{3/p}(\omega))}$ carefully. As in [6], we have

$$\sum_{j>M} 2^{j\frac{3}{p}}\|\Delta_j a_0\|_{L^p} \le C2^{-\frac{1}{2}M}\|a_0\|_{H^2},$$

and

$$\sum_{j\le M} 2^{j\frac{3}{p}}\omega_j(T)\|\Delta_j a_0\|_{L^p} \le C2^{2M}T^{\frac{1}{2}}\|a_0\|_{H^2} + C2^{-\frac{1}{2}M}\|a_0\|_{H^2}.$$

So if we take M large enough and

$$(3.24) \qquad\qquad T = \frac{c}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^4(1+\|a_0\|_{H^2})^{12}},$$

we have

$$(3.25) \qquad\qquad \|a_0\|_{\dot{B}_p^{3/p}(\omega)} \le c',$$

where c, c' are small enough constants. Similar to (3.22), we also have

$$(3.26) \qquad \|a\|_{\tilde{L}_T^\infty(\dot{B}_p^{3/p}(\omega))} \le \|a_0\|_{\dot{B}_p^{3/p}(\omega)} + \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^4}.$$

In the following proposition, we show that this solution allows us to propagate the regularity of the initial data in Sobolev space with low regularity.

Proposition 3.3. Let $p \in (3,6)$ and $1-\frac{3}{p} < \delta < \frac{3}{p}$. Assume that (ρ, u, B) is a solution for system (1.1). If $(a_0, u_0, B_0) \in H^{1-\delta} \times \dot{H}^{-\delta} \times \dot{H}^{-\delta}$, then

$$\|a\|_{\tilde{L}_T^\infty(\dot{B}_{2,2}^{1-\delta})} \le C\left(\|a_0\|_{H^{1-\delta}} + \|u_0\|_{\dot{H}^{-\delta}} + \|B_0\|_{\dot{H}^{-\delta}}\right),$$

$$(3.27) \qquad \|(u,B)\|_{\tilde{L}_T^\infty(\dot{B}_{2,2}^{-\delta})} + \|(u,B)\|_{\tilde{L}_T^1(\dot{B}_{2,2}^{2-\delta})}$$

$$\le C\left(1 + \|a_0\|_{\dot{B}_p^{3/p}}\right)\left(\|(u_0,B_0)\|_{\dot{H}^{-\delta}} + TP_+\|a_0\|_{H^{1-\delta}}\right).$$

Proof. The proof can be divided into three steps. Set $\beta = 1-\delta$. Without loss of generality, we may assume that c, c_2, c_3, c', T (choose as in (3.25) and (3.24)) small enough such that

$$(3.28) \qquad \begin{aligned} &C\frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^5} + 2C\left(c' + \frac{c_2+c_3}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^4}\right) \\ &+ C\frac{2c_2}{(1+\|a_0\|_{\dot{B}_p^{3/p}})} + C\frac{2c_2}{(1+\|a_0\|_{\dot{B}_p^{3/p}})^2} + CTP_+ \le \frac{1}{2}, \end{aligned}$$

where C is the constant appearing in the following estimates.
Step 1 : Estimate for the Transport Equation. This estimate is similar to the Navier-Stokes system, so we omit the details and only give the results as follows

$$(3.29) \qquad \|a\|_{\tilde{L}_T^\infty(\dot{B}_{2,2}^\beta)} \le C\left(\|a_0\|_{\dot{B}_{2,2}^\beta} + \|u\|_{\tilde{L}_T^1(\dot{B}_{2,2}^{\beta+1})}\right).$$

For details, we give the reference [6] on page 192.
Step 2 : Estimate for the Momentum Equation. Denote $\bar{\mu} = \frac{\mu}{\rho}$ and $\bar{\lambda} = \frac{\lambda}{\rho}$,

apply the operator Δ_j to the momentum equation of (1.1), we can obtain

$$\partial_t \Delta_j u - \text{div}(\bar{\mu}\nabla\Delta_j u) - \nabla(\bar{\lambda}\text{div}\Delta_j u)$$

(3.30)
$$= \Delta_j G + \text{div}([\Delta_j, \bar{\mu}]\nabla u) + \nabla([\Delta_j, \bar{\lambda}]\text{div}u)$$

$$+ \Delta_j\left(\frac{1}{\rho}B\cdot\nabla B\right) - \frac{1}{2}\Delta_j\left(\frac{1}{\rho}\nabla|B|^2\right),$$

where

$$G = -u\cdot\nabla u - \frac{\bar{\rho}P'(\rho)}{\rho}\nabla u + \frac{\mu}{\rho^2}\nabla\rho\nabla u + \frac{\lambda}{\rho^2}\nabla\rho\text{div}u.$$

Taking L^2 energy estimate for weighted Besov space, we have

(3.31)
$$\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + \|u\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})} \leq C(\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + \|G\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2}(\omega))})$$

$$+ C\left\|2^{j\beta}\omega_j(T)\left(\|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_T(L^2)} + \|[\Delta_j, \bar{\lambda}]\nabla u\|_{L^1_T(L^2)}\right)\right\|_{\ell^2}$$

$$+ C\left\|\frac{1}{\rho}B\cdot\nabla B\right\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} + C\left\|\frac{1}{\rho}\nabla(|B|^2)\right\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})}$$

Similar to the Navier-Stokes system, we have

(3.32)
$$\left\|2^{j\beta}\omega_j(T)\left(\|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_T(L^2)} + \|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_T(L^2)}\right)\right\|_{\ell^2}$$

$$\leq C\|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p(\omega))}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})}$$

and

(3.33)
$$\|G\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2}(\omega))} \leq C\big(\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})}$$

$$+ \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p(\omega))}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + TP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^{\beta}_{2,2})}\big).$$

Then, we estimate the term which is not appeared in the Navier-Stokes system as follows

(3.34)
$$\left\|\frac{1}{\rho}B\cdot\nabla B\right\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} \leq C(1 + \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)})\|B\cdot\nabla B\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})}$$

$$\leq C(1 + \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)})\|B\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})}.$$

Plugging estimates (3.32), (3.33) and (3.34) into (3.31), we obtain

(3.35)
$$\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + \|u\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})} \leq C\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + C\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})}$$

$$+ C\|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p(\omega))}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + CTP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^{\beta}_{2,2})}$$

$$+ C(1 + \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)})\|B\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_T(\dot{B}^{\beta}_{2,2})}.$$

Taking L^2 energy estimates for equation (3.30), we obtain

(3.36)
$$\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} \leq C\big(\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + \|G\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})}\big)$$

$$+ C\left\|2^{j\beta}\left(\|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_T(L^2)} + \|[\Delta_j, \bar{\lambda}]\nabla u\|_{L^1_T(L^2)}\right)\right\|_{\ell^2}$$

$$+ C\left\|\frac{1}{\rho}B\cdot\nabla B\right\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} + C\left\|\frac{1}{\rho}\nabla(|B|^2)\right\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})}.$$

The first two terms can be estimated as in the Navier-Stokes system, we only give the results as follows

$$(3.37) \quad \left\| 2^{j\beta} \left(\|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_T(L^2)} + \|[\Delta_j, \bar{\lambda}]\nabla u\|_{L^1_T(L^2)} \right) \right\|_{\ell^2}$$
$$\leq C\|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})},$$

$$(3.38) \quad \|G\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} \leq C\big(\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})} + \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})}$$
$$+ TP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^\beta_{2,2})} \big).$$

For the terms not appeared in the Navier-Stokes system, we can estimate as (3.34). Plugging (3.37), (3.38) and (3.34) into (3.36), we have

$$(3.39) \quad \|u\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} \leq C\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + C\|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})}$$
$$+ C\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})} + CTP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^\beta_{2,2})}$$
$$+ C(1 + \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_p)})\|B\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})}.$$

Step 3 : Estimate for the Magnetic Field Equation. Applying the operator Δ_j to the third equation of system (1.1), we obtain

$$\partial_t \Delta_j B - \nu \Delta \Delta_j B = -\Delta_j \text{div}(Bu^T - uB^T).$$

Performing L^2 energy estimates, we could have

$$(3.40) \quad \|B\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} + \|B\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} \leq C\|B_0\|_{\dot{B}^{\beta-1}_{2,2}}$$
$$+ C\|\text{div}(Bu^T - uB^T)\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})}.$$

Since

$$(3.41) \quad \|u \cdot \nabla B\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} \leq C\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|\nabla B\|_{\tilde{L}^2_T(\dot{B}^{\beta-1}_{2,2})}$$
$$\leq C\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})},$$

and

$$(3.42) \quad \|B \cdot \nabla u\|_{\tilde{L}^1_T(\dot{B}^{\beta-1}_{2,2})} \leq C\|B\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})},$$

we know that

$$(3.43) \quad \|B\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} + \|B\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} \leq C\|B_0\|_{\dot{B}^{\beta-1}_{2,2}} + C\|u\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})}$$
$$+ C\|B\|_{\tilde{L}^2_T(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})}.$$

Combing estimates (3.35) and (3.43), we will get

$$(3.44) \quad \|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + \|u\|_{\tilde{L}^2_T(\dot{B}^\beta_{2,2})} + \|B\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})}$$
$$\leq C\big(\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + \|B_0\|_{\dot{B}^{\beta-1}_{2,2}} + TP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^\beta_{2,2})} \big).$$

Combining estimates (3.39) and (3.43), we will have

$$(3.45) \quad \|u\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} + \|B\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})}$$
$$\leq C\big(\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + \|B_0\|_{\dot{B}^{\beta-1}_{2,2}} + TP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^\beta_{2,2})} \big)$$
$$+ C\|a_0\|_{\dot{B}^{3/p}_p}\|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})}.$$

At this stage, combining (3.44) and (3.45), we obtain

$$
(3.46) \quad
\begin{aligned}
&\|u\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} + \|B\|_{\tilde{L}^\infty_T(\dot{B}^{\beta-1}_{2,2})} + \|u\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} + \|B\|_{\tilde{L}^1_T(\dot{B}^{\beta+1}_{2,2})} \\
&\qquad \leq C\big(1 + \|a_0\|_{\dot{B}^{3/p}_p}\big)\big(\|u_0\|_{\dot{B}^{\beta-1}_{2,2}} + \|B_0\|_{\dot{B}^{\beta-1}_{2,2}} + TP_+\|a\|_{\tilde{L}^\infty_T(\dot{B}^\beta_{2,2})}\big).
\end{aligned}
$$

At last, combining (3.29) and (3.46), we finally get the desired results (3.27). $\qquad\square$

In the last part of this section, we prove the solution propagate the regularity of the initial data in Sobolev space with high regularity.

Proposition 3.4. Assume that (ρ, u, B) is a solution of system (1.1) on $[0, T]$, which satisfies $\rho \geq c_0$,

$$
(3.47) \qquad a \in \tilde{L}^\infty_T(\dot{B}^{3/p}_p), \quad u, B \in \tilde{L}^\infty_T(\dot{B}^{3/p}_p) \cap \tilde{L}^1_T(\dot{B}^{3/p+1}_p),
$$

where $a = \rho - 1$. If $(a_0, u_0, B_0) \in H^s \times H^{s-1} \times H^{s-1}$ for $s \geq 3$, then we have

$$
(3.48) \qquad a \in \tilde{L}^\infty_T(H^s), \quad u, B \in \tilde{L}^\infty_T(H^{s-1}) \cap \tilde{L}^1_T(H^{s+1}).
$$

Proof. Considering (3.47), we can divide the time interval $[0, T]$ into finitely many small intervals $[T_i, T_{i+1}]$ with $i = 0, 1, \ldots, N$ such that

$$
(3.49) \quad
\begin{aligned}
&T_{i+1} - T_i \leq \epsilon, \quad \|a\|_{\tilde{L}^\infty(T_i, T_{i+1}; \dot{B}^{3/p}_p(\omega^i))} \leq \epsilon, \\
&\|(u, B)\|_{\tilde{L}^1(T_i, T_{i+1}; \dot{B}^{3/p+1}_p)} + \|(u, B)\|_{\tilde{L}^2(T_i, T_{i+1}; \dot{B}^{3/p}_p)} \leq \epsilon,
\end{aligned}
$$

for some ϵ small enough. Here $\omega^i = \{\omega^i_k\}$ stands for

$$
\omega^i_k = \sum_{\ell \geq k} 2^{k-\ell}\big(1 - e^{-c2^{2\ell(T_{i+1}-T_i)}}\big)^{\frac{1}{2}}.
$$

Denote $\tilde{L}^q_{T_i}(\dot{B}^s_{p,r}) := \tilde{L}^q(T_i, T_{i+1}; \dot{B}^s_{p,r})$. For the density, the estimate is similar to the Navier-Stokes equations, so we omit the details

$$
(3.50) \qquad \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})} \leq C\big(\|a(T_i)\|_{\dot{B}^s_{2,2}} + \|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})}\big).
$$

For the velocity field, as in the proof of Proposition 3.3, we have

$$
(3.51) \quad
\begin{aligned}
&\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} \\
&\quad \leq C\big(\|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|G\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2}(\omega^i))}\big) \\
&\qquad + C\left\|2^{js}\omega_j(t)\left(\|[\Delta_j, \bar{\lambda}]\nabla u\|_{L^1_{T_i}(L^2)} + \|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_{T_i}(L^2)}\right)\right\|_{\ell^2} \\
&\qquad + C\left\|\frac{1}{\rho}B\cdot\nabla B\right\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})} + C\left\|\frac{1}{\rho}\nabla(|B^2|)\right\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})}.
\end{aligned}
$$

Similar to the Navier-Stokes system, we have

$$
(3.52) \quad
\begin{aligned}
\|G\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2}(\omega^i))} &\leq C\big(\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} \\
&\quad + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p(\omega^i))}\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{3/p+1}_p)} \\
&\quad + (T_{i+1} - T_i)\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}\big),
\end{aligned}
$$

and

$$
(3.53) \quad
\begin{aligned}
&\left\|2^{js}\omega^i_j(t)\left(\|[\Delta_j, \bar{\lambda}]\nabla u\|_{L^1_{T_i}(L^2)} + \|[\Delta_j, \bar{\mu}]\nabla u\|_{L^1_{T_i}(L^2)}\right)\right\|_{\ell^2} \\
&\qquad \leq C\big(\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p(\omega^i))}\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{3/p+1}_p)}\big).
\end{aligned}
$$

For the last two terms in 3.51, all can be estimated as follows

$$
\left\|\frac{1}{\rho}B \cdot \nabla B\right\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})} \le C\|B \cdot \nabla B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})} + C\left\|\frac{a}{1+a}B \cdot \nabla B\right\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})}
$$

$$
\le C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + C\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p)}\|B \cdot \nabla B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})}
$$

(3.54)
$$
+ C\|B \cdot \nabla B\|_{\tilde{L}^1_{T_i}(\dot{B}^{3/p-1}_p)}\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}
$$

$$
\le C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + C\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

$$
+ C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}.
$$

Plugging (3.52), (3.53) and (3.54) into (3.51), we will obtain

$$
\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

$$
\le C\big(\|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}(\|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{3/p}_p)} + (T_{i+1} - T_i)))
$$

(3.55)
$$
+ C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + C\|a_0\|_{\dot{B}^{3/p}_p}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

$$
+ C\|B\|^2_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})}.
$$

For the equation of magnetic field, using similar methods as in Proposition 3.3, we obtain

(3.56)
$$
\|B\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{s-1}_{2,2})} + \|B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} \le C\|B_0\|_{\dot{B}^{s-1}_{2,2}} + C\|u \cdot \nabla B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})}
$$

$$
+ C\|B \cdot \nabla u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})}.
$$

Since

$$
\|u \cdot \nabla B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})} \le C(\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|\nabla B\|_{\tilde{L}^2_{T_i}(\dot{B}^{s-1}_{2,2})} + \|\nabla B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p-1}_p)}\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})})
$$

$$
\le C\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})},
$$

and

$$
\|B \cdot \nabla u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s-1}_{2,2})} \le C(\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + \|u\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}),
$$

we know that

$$
\|B\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{s-1}_{2,2})} + \|B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

(3.57)
$$
\le C\|B(T_i)\|_{\dot{B}^{s-1}_{2,2}} + C\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

$$
+ C\|B\|_{\tilde{L}^2_{T_i}(\dot{B}^{3/p}_p)}\|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}.
$$

Combining (3.50), (3.55) and (3.57), we have

$$
\|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})} + \|u\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|u\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})} + \|B\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} + \|B\|_{\tilde{L}^2_{T_i}(\dot{B}^s_{2,2})}
$$

(3.58)
$$
\le C(\|a(T_i)\|_{\dot{B}^s_{2,2}} + \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
$$

Estimate velocity as in (3.36) with regularity index to be $s-1$ and combine the estimate about magnetic field B, we finally get

$$
\|u\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{s-1}_{2,2})} + \|B\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{s-1}_{2,2})} \le C(1 + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p)})(\|a(T_i)\|_{\dot{B}^s_{2,2}}
$$

(3.59)
$$
+ \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
$$

Combining (3.58) and (3.59), we arrive at

$$
\begin{aligned}
(3.60) \quad & \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^s_{2,2})} + \|(u,B)\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{s-1}_{2,2})} + \|(u,B)\|_{\tilde{L}^1_{T_i}(\dot{B}^{s+1}_{2,2})} \\
& \leq C(1 + \|a\|_{\tilde{L}^\infty_{T_i}(\dot{B}^{3/p}_p)})(\|a(T_i)\|_{\dot{B}^s_{2,2}} + \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
\end{aligned}
$$

By induction, we know that

$$
\begin{aligned}
(3.61) \quad & \|a\|_{\tilde{L}^\infty_{T}(\dot{B}^s_{2,2})} + \|(u,B)\|_{\tilde{L}^\infty_{T}(\dot{B}^{s-1}_{2,2})} + \|(u,B)\|_{\tilde{L}^1_{T}(\dot{B}^{s+1}_{2,2})} \\
& \leq C(1 + \|a\|_{\tilde{L}^\infty_{T}(\dot{B}^{3/p}_p)})^{N+1}(\|a(T_i)\|_{\dot{B}^s_{2,2}} + \|u(T_i)\|_{\dot{B}^{s-1}_{2,2}} + \|B(T_i)\|_{\dot{B}^{s-1}_{2,2}}).
\end{aligned}
$$

Hence, we complete the proof. $\qquad\square$

4. Hoff's Energy Method

In two papers [15, 16], D. Hoff construct a global weak solution for Navier-Stokes equations with discontinuous initial data with small energy. In the paper [14], A. Suen and D. Hoff generalize the results for Navier-Stokes equations to compressible MHD system. Here, we use the idea to our case. Comparing to [14], we remove the restriction on viscosity for we have stronger condition on the initial data.

We set $\sigma(t) := \min(1, t)$, define

$$
A_1(T) = \sup_{0 \leq t \leq T} (\|\nabla u(t)\|^2_{L^2} + \|\nabla B(t)\|^2_{L^2}) + \int_0^T \int_{\mathbb{R}^3} \rho|\dot{u}|^2 + |B_t|^2 \, dxdt,
$$

$$
\begin{aligned}
A_2(T) = {} & \sup_{0 \leq t \leq T} \left(\sigma(t) \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \, dx + \sigma(t) \int_{\mathbb{R}^3} |B_t|^2 \, dx \right) \\
& + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla \dot{u}|^2 \, dxdt + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla B_t|^2 \, dxdt,
\end{aligned}
$$

and

$$
E(T) = \int_{\mathbb{R}^3} \sigma(t) \left(|\nabla B|^2|B|^2 + |\nabla B|^2|u|^2 + |\nabla u|^2|B|^2 \right) \, dx,
$$

$$
\begin{aligned}
H(T) = {} & \int_0^T \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt \\
& + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla B|^4 \, dxdt + \int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla u|^4 \, dxdt.
\end{aligned}
$$

In the following sections, we denote

$$
\dot{f} = f_t + u \cdot \nabla f, \quad \mu' = \lambda + \mu,
$$

and

$$
C_0 = \|\rho_0 - \bar{\rho}\|^2_{L^2} + \|u_0\|^2_{H^2} + \|B_0\|^2_{H^1}.
$$

Throughout this section, we denote by C a constant depending only on $\lambda, \mu, c_0, \bar{\rho}, P_+$ and P_-^{-1}.

Theorem 4.1. *Let (ρ, u, B) be a solution of system (1.1) satisfying*

$$
\rho - \bar{\rho} \in C([0,T]; H^2), \quad u, B \in C([0,T]; H^2) \cap L^2(0,T; H^3).
$$

Then, there exists a constant ϵ_0 depending only on $\nu, \lambda, \mu, c_0, \bar{\rho}, P_+, P_-^{-1}$ such that if the initial data (ρ_0, u_0, B_0) satisfies

$$c_0 \le \rho_0(x) \le c_0^{-1}, \quad x \in \mathbb{R}^3,$$
$$\|\rho - \bar{\rho}\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|B_0\|_{H^1}^2 \le \epsilon_0,$$

then we have

$$\frac{c_0}{2} \le \rho(t, x) \le 2c_0^{-1}, \quad (t, x) \in [0, T] \times \mathbb{R}^3,$$
$$A(T) := A_1(T) + A_2(T) \le \epsilon_0^{\frac{1}{2}}.$$

Proof. Considering the assumption, there exists a $0 < T_0 \le T$ such that the solution (ρ, u, B) satisfies

$$\frac{c_0}{2} \le \rho(t, x) \le 2c_0^{-1}, \quad (t, x) \in [0, T_0] \times \mathbb{R}^3,$$
$$A_1(T_0) + A_2(T_0) \le \epsilon_0^{\frac{1}{2}}.$$

Without loss of generality, we assume that T_0 is a maximal time so that the above inequalities hold. In the following, we will give a refined estimates on $[0, T]$ for the solution. Due to the proof is too long, we divide the proof into several lemmas.

Lemma 4.2. *(L^2 energy estimate)*

$$\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 + \rho|u|^2 + |B|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla B|^2 \, dx dt \le CC_0.$$

The proof of this lemma is exactly the same as Lemma 2.2 in [19], so we omit it.

Lemma 4.3. *(H^1 energy estimate)*

$$A_1(T_0) \le CC_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \, dx dt$$
$$+ C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dx dt.$$

Proof. Multiply the second equation in (1.1) by $\dot{u}$ and integrate over $\mathbb{R}^3$, we will obtain

(4.1)
$$\int_{\mathbb{R}^3} \rho|\dot{u}|^2 \, dx = \int_{\mathbb{R}^3} (-\dot{u} \cdot \nabla P + \mu \Delta u \, \dot{u} + \mu' \nabla \mathrm{div} u \, \dot{u}$$
$$- \nabla(\frac{1}{2}|B|^2)\dot{u} + \mathrm{div}(B\, B^T)\, \dot{u}) \, dx.$$

By the continuity equation, we have

(4.2)
$$\partial P + \mathrm{div}(uP) = \mathrm{div} u\,(P - P'\rho).$$

There are some terms can be estimated as in the Navier-Stokes equations, so we omit the details and only give the results as follows

(4.3)
$$\int_{\mathbb{R}^3} -\dot{u} \cdot \nabla P \, dx \le \partial_t \int_{\mathbb{R}^3} \mathrm{div} u\,(P - P(\bar{\rho})) \, dx + C\|\nabla u\|_{L^2}^2,$$

(4.4)
$$\mu \int_{\mathbb{R}^3} \Delta u \, \dot{u} \, dx \le -\frac{\mu}{2}\partial_t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla u|^3 \, dx,$$

$$(4.5) \qquad \mu' \int_{\mathbb{R}^3} \nabla \mathrm{div} u \, \dot{u} \, dx \leq -\frac{\lambda}{2} \partial_t \int_{\mathbb{R}^3} |\mathrm{div} u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla u|^3 \, dx,$$

and

$$(4.6) \qquad \begin{aligned} \int_{\mathbb{R}^3} \mathrm{div} u \, (P - P(\bar{\rho})) \, dx &\leq \|\mathrm{div} u\|_{L^2} \|P(\rho) - P(\bar{\rho})\|_{L^2} \\ &\leq C C_0^{\frac{1}{2}} \|\mathrm{div} u\|_{L^2}. \end{aligned}$$

The following terms are not appeared in Navier-Stokes system

$$(4.7) \qquad \begin{aligned} \int_{\mathbb{R}^3} -\nabla(\frac{1}{2}|B|^2) \, \dot{u} \, dx + \int_{\mathbb{R}^3} \mathrm{div}(B \, B^T) \, \dot{u} \, dx &\leq C \int_{\mathbb{R}^3} |\nabla B||B||\dot{u}| \, dx \\ &\leq C \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \, dx + \epsilon \int_{\mathbb{R}^3} |\dot{u}|^2 \, dx. \end{aligned}$$

Multiply the third equation in (1.1) with B_t and integrate over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} |B_t|^2 \, dx + \int_{\mathbb{R}^3} \mathrm{div}(B \, u^T - u \, B^T) \, B_t \, dx = -\frac{\nu}{2} \partial_t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx.$$

Integrate the above equality from 0 to T_0, we will obtain

$$(4.8) \qquad \begin{aligned} \frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla B|^2 \, dx &+ \int_0^{T_0} \int_{\mathbb{R}^3} |B_t|^2 \, dx \\ &\leq C C_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \, dx dt. \end{aligned}$$

Combining (4.1), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), we finally get the desired result. □

Lemma 4.4. *(H^2 energy estimate)*

$$\begin{aligned} A_2(T_0) \leq &C C_0 + C A_1(T_0) + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx dt \\ &+ C \int_0^T \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2 + |u|^2(|\nabla B|^2 + |\nabla u|^2)) \, dx dt. \end{aligned}$$

Proof. Take material derivative to the second equation of (1.1) to obtain

$$\begin{aligned} \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla P_t + \mathrm{div}(\nabla P \otimes u) + \nabla(\frac{1}{2}|B|^2)_t + \mathrm{div}(u \cdot \nabla(\frac{1}{2}|B|^2)) \\ - \mathrm{div}(B \, B^T)_t - \mathrm{div}(u \, \mathrm{div}(B \, B^T)) = \mu \Delta u_t + \mu \mathrm{div}(u \, \Delta u) \\ + \mu' \nabla \mathrm{div} u_t + \mu' \mathrm{div}(u \cdot \nabla \mathrm{div} u). \end{aligned}$$

Multiply $\sigma(t)\dot{u}$ on both sides of the above equality, we will have

$$
\begin{aligned}
\partial_t \left(\frac{\sigma}{2} \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \, dx \right) - \frac{1}{2}\sigma' \int_{\mathbb{R}^3} \rho|\dot{u}|^2 \, dx &= - \int_{\mathbb{R}^3} \sigma \dot{u}^j (\partial_j P_t + \operatorname{div}(\partial_j P \, u)) \, dx \\
&= - \int_{\mathbb{R}^3} \sigma \dot{u}^j ((\frac{1}{2}|B|^2)_{x_j,t} + \operatorname{div}(u \, \partial_j(\frac{1}{2}|B|^2))) \, dx \\
&\quad + \int_{\mathbb{R}^3} \sigma \dot{u}^j (\operatorname{div}(B^j B)_t + \operatorname{div}(u\operatorname{div}(B^j B))) \, dx \\
&\quad + \mu\sigma \int_{\mathbb{R}^3} \dot{u} \, (\Delta u_t + \operatorname{div}(\Delta u \otimes u)) \, dx \\
&\quad + \mu'\sigma \int_{\mathbb{R}^3} \dot{u} \, (\nabla\operatorname{div}u_t + \operatorname{div}(\nabla\operatorname{div}u \otimes u)) \, dx.
\end{aligned}
\tag{4.9}
$$

Similar to the H^1 estimate, there are some terms same as Navier-Stokes equations and we only give the results as follows

$$
- \int_{\mathbb{R}^3} \sigma \dot{u}^j (\partial_j P_t + \operatorname{div}(\partial_j P \, u)) \, dx \leq C\sigma \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2},
\tag{4.10}
$$

$$
\mu\sigma \int_{\mathbb{R}^3} \dot{u} \, (\Delta u_t + \operatorname{div}(\Delta u \otimes u)) \, dx \leq \sigma \int_{\mathbb{R}^3} -\frac{3\mu}{4}|\nabla \dot{u}|^2 \, dx + \sigma C \int_{\mathbb{R}^3} |\nabla u|^4 \, dx,
\tag{4.11}
$$

$$
\begin{aligned}
\mu'\sigma \int_{\mathbb{R}^3} \dot{u} \, (\nabla\operatorname{div}u_t + \operatorname{div}(\nabla\operatorname{div}u \otimes u)) \, dx \\
\leq \sigma \int_{\mathbb{R}^3} -\frac{\lambda}{2}|\operatorname{div}\dot{u}|^2 + \frac{\mu}{4}|\nabla \dot{u}|^2 + C|\nabla u|^4 \, dx.
\end{aligned}
\tag{4.12}
$$

For the terms not appeared in the Navier-Stokes equations, we have

$$
\begin{aligned}
- \int_{\mathbb{R}^3} \sigma \dot{u}^j (\partial_j(\frac{1}{2}|B|^2)_t + \operatorname{div}(u \, \partial_j(\frac{1}{2}|B|^2))) \, dx \\
\leq \epsilon \int_{\mathbb{R}^3} \sigma|\nabla \dot{u}|^2 \, dx + C \int_{\mathbb{R}^3} \sigma|B|^2(|B_t|^2 + |u|^2|\nabla B|^2) \, dx.
\end{aligned}
\tag{4.13}
$$

For the equation about magnetic field, multiplying σB_t on both sides, we will have

$$
\begin{aligned}
\frac{1}{2}\sigma \int_{\mathbb{R}^3} |B_t|^2 \, dx + \nu \int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla B_t|^2 \, dxdt \\
= - \int_0^{T_0} \int_{\mathbb{R}^3} \sigma B_t(\operatorname{div}(B \, u^T - u \, B^T))_t \, dxdt + \frac{1}{2} \int_0^{T_0} \int_{\mathbb{R}^3} \sigma'|B_t|^2 \, dxdt.
\end{aligned}
\tag{4.14}
$$

For the first term on the right hand side of the above equality, we have

$$-\int_0^{T_0} \int_{\mathbb{R}^3} \sigma B_t (\operatorname{div}(B\, u^T - u\, B^T))_t \, dxdt$$

$$= \int_0^{T_0} \int_{\mathbb{R}^3} \sigma \nabla B_t (B_t\, u^T + B\, u_t^T - u_t\, B^T - u\, B_t^T) \, dxdt$$

$$(4.15) \qquad \leq \epsilon \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B_t|^2 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|B_t|^2 |u|^2 + |B|^2 |u_t|^2) \, dxdt$$

$$\leq \epsilon \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B_t|^2 \, dxdt$$

$$\qquad + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|B_t|^2 |u|^2 + |B|^2 |\dot{u}|^2 + |B|^2 |u|^2 |\nabla u|^2) \, dxdt,$$

where ϵ is a small enough positive number. Combining estimates (4.9), (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), we will obtain

$$\sigma \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + |B_t|^2 \, dx + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|\nabla \dot{u}|^2 + |\nabla B_t|^2) \, dxdt$$

$$\leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|_{L^2} |\nabla \dot{u}|_{L^2} \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dxdt$$

$$(4.16)$$

$$\qquad + C \int_0^{1\wedge T_0} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \, dxdt + C \int_0^{1\wedge T_0} \int_{\mathbb{R}^3} |B_t|^2 \, dxdt$$

$$\qquad + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2 + |u|^2(|\nabla B|^2 + |\nabla u|^2)) \, dxdt.$$

We can easily obtain the following estimates

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|_{L^2} |\nabla \dot{u}|_{L^2} \, dxdt$$

$$(4.17)$$

$$\qquad \leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 \, dxdt$$

$$\qquad \leq C C_0 + C A_1(T_0),$$

$$(4.18) \qquad \int_0^{1\wedge T_0} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \, dxdt + \int_0^{1\wedge T_0} \int_{\mathbb{R}^3} |B_t|^2 \, dxdt \leq C A_1(T_0).$$

Substitute (4.17) and (4.18) into (4.16), we finally complete the proof. $\qquad\square$

Lemma 4.5.

$$A_1(T_0) + A_2(T_0) \leq C C_0 + C C_0 (C_0^{5/3} + C_0 A_2(T_0)) A_1(T_0)^{3/2}$$

$$\qquad + C C_0 A_1(T_0)^2 + C C_0^2 A_1(T_0)^{3/2} (\sigma |\nabla^2 B|_{L^2}^2)$$

$$\qquad + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 + |\nabla B|^3 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|\nabla u|^4 + |\nabla B|^4) \, dxdt.$$

Proof. Combining Lemma 4.2, Lemma 4.3 and Lemma 4.4, we will have

$$A_1(T_0) + A_2(T_0) \leq CC_0 + C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt$$

(4.19)
$$+ C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2 + |u|^2(|\nabla B|^2 + |\nabla u|^2)) dxdt$$

$$+ C \int_0^{T_0} \int_{\mathbb{R}^3} (|B|^2 + |u|^2)(|\nabla B|^2 + |\nabla u|^2) \, dxdt.$$

Next we need to estimate some typical terms on the right hand side.
Term 1 : $\int_0^{T_0} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dxdt$

$$\int_0^{T_0} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dxdt \leq \int_0^{T_0} \left(\int_{\mathbb{R}^3} |B|^6 \, dx \right)^{1/3} \left(\int_{\mathbb{R}^3} |\nabla B|^3 \, dx \right)^{2/3} dt$$

$$\leq C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dxdt$$

$$\leq C \int_0^{T_0} \left(\int_{\mathbb{R}^3} |\nabla B|^3 \, dx \right)^3 dt + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt$$

$$\leq C \left(\int_{\mathbb{R}^2} |\nabla B|^2 \, dx \right)^2 \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 \, dxdt + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt$$

$$\leq CC_0 A_1(T_0)^2 + C \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt.$$

Term 2 : $\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |B_t|^2 \, dxdt$

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |B_t|^2 \, dxdt \leq \left(\int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dxdt \right)^{1/3} \left(\int_0^{T_0} \int_{\mathbb{R}^3} \sigma^{3/2} |B_t|^3 \, dxdt \right)^{2/3}$$

$$\leq \left(\int_0^{T_0} \left(\int_{\mathbb{R}^3} |\nabla B|^2 \, dx \right)^3 dt \right)^{1/3} \left(\sigma \|B_t\|_{L^2}^2 \right)^{1/3} \left(\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B_t|^2 \, dxdt \right)^{2/3}$$

$$\leq CC_0^{1/3} A_1(T_0)^{4/3} A_2(T_0)^{1/3}.$$

Term 3 : $\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |u|^2 |\nabla B|^2 \, dxdt$

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |B|^2 |u|^2 |\nabla B|^2 \, dxdt \leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|B|^8 + |u|^8 + |\nabla B|^4) \, dxdt$$

$$\leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dxdt + (\|u\|_{L^4}^4 + \|B\|_{L^4}^4) \int_0^{T_0} \sigma(\|u\|_{L^\infty}^4 + \|B\|_{L^\infty}^4) \, dt$$

Since

$$\|u\|_{L^4}^4 + \|B\|_{L^4}^4 \leq \|u\|_{L^2} \|\nabla u\|_{L^2}^3 + \|B\|_{L^2} \|\nabla B\|_{L^2}^3,$$

and

$$\int_0^{T_0} \sigma\|u\|_{L^\infty}^4 + \sigma\|B\|_{L^\infty}^4 \, dt \leq \int_0^{T_0} \sigma\|u\|_{L^6}^2\|\nabla u\|_{L^6}^2 + \sigma\|B\|_{L^6}^2\|\nabla B\|_{L^6}^2 \, dt$$

$$\leq \int_0^{T_0} \sigma\|\nabla u\|_{L^2}^2(\|\rho\dot{u}\|_{L^2} + \|P(\rho) - P(\bar{\rho})\|_{L^6})^2 \, dt + \int_0^{T_0} \sigma\|\nabla B\|_{L^2}^2\|\nabla^2 B\|_{L^2}^2 \, dt$$

$$\leq \sigma\|\rho\dot{u}\|_{L^2}^2 \int_0^{T_0} \|\nabla u\|_{L^2}^2 \, dt + CC_0^{5/3} + CC_0\sigma\|\nabla^2 B\|_{L^2}^2,$$

we can get the estimate

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma|B|^2|u|^2|\nabla B|^2 \, dxdt \leq \int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla B|^4 \, dxdt$$

$$+ CC_0 A_1(T_0)^{3/2}(C_0^{5/3} + C_0\sigma\|\nabla^2 B\|_{L^2}^2 + C_0 A_2(T_0)).$$

where we used Lemma 4.2 and the definition of A_1, A_2.

Using similar procedure as did for Term 1, we have

$$(4.20) \qquad \int_0^{T_0} \int_{\mathbb{R}^3} (|B|^2 + |u|^2)(|\nabla B|^2 + |\nabla u|^2) \, dxdt$$

$$\leq CC_0 A_1(T_0)^2 + C\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt + C\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt.$$

Using similar procedure as did for Term 2, we have

$$(4.21) \qquad \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2) \, dxdt \leq CC_0^{1/3} A_1(T_0)^{4/3} A_2(T_0)^{1/3}.$$

Using similar procedure as did for Term 3, we have

$$(4.22) \qquad \int_0^{T_0} \int_{\mathbb{R}^3} \sigma(|u|^2 + |B|^2)|u|^2(|\nabla B|^2 + |\nabla u|^2) \, dxdt$$

$$\leq CC_0 A_1(T_0)^{3/2}(C_0^{5/3} + C_0\sigma\|\nabla^2 B\|_{L^2}^2 + C_0 A_2(T_0))$$

$$+ \int_0^{T_0} \int_{\mathbb{R}^3} \sigma|\nabla B|^4 \, dxdt.$$

At last substitute (4.20), (4.21), (4.22) into (4.19) and do simple reduction, we will complete our proof. $\square$

Before going to the following part, we need to introduce some notations. Let

$$(4.23) \qquad F = (\lambda + \mu)\mathrm{div}u - P(\rho) + P(\bar{\rho}),$$

and

$$(4.24) \qquad \omega^{j,k} = u_{x_k}^j - u_{x_j}^k, \quad j,k = 1,2,3.$$

Through simple calculations, we could find that

$$(4.25) \qquad \mu\Delta\omega^{j,k} = (\rho\dot{u}^j)_{x_k} - (\rho\dot{u}^k)_{x_j} - (B\cdot\nabla B^j)_{x_k} + (B\cdot\nabla B^k)_{x_j},$$

and

$$(4.26) \qquad \Delta F = \mathrm{div}g$$

where $g^j = \rho\dot{u}^j + (\frac{1}{2}|B|^2)_{x_j} - \mathrm{div}(B^j B)$ with $j = 1,2,3$.

Lemma 4.6.

$$E(T_0) \leq CC_0 + P_E(A(T_0)),$$

where $A(T_0) = A_1(T_0) + A_2(T_0)$ and P_E is a polynomial function with 2 as the lowest order.

Proof. For the term $\int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx$, we have

$$(4.27) \qquad \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \leq \sigma \|u\|_{L^\infty}^2 \int_{\mathbb{R}^2} |\nabla B|^2 \, dx$$
$$\leq CA_1(T_0) \left(\sigma \|u\|_{L^4}^2 + \sigma \|\nabla u\|_{L^4}^2 \right).$$

For term $\sigma \|u\|_{L^4}^2$ which appeared in the above inequality, we have

$$(4.28) \qquad \sigma \|u\|_{L^4}^2 \leq \sigma \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{3/2} \leq CC_0^{1/2} A_1(T_0)^{3/4}.$$

The analysis about $\sigma \|\nabla u\|_{L^4}^2$ is a little bit more complex, by (4.23), we know that

$$(4.29) \qquad \sigma \|\nabla u\|_{L^4}^2 \leq \sigma \|F\|_{L^4}^2 + \sigma \|\omega\|_{L^4}^2 + \sigma \|P(\rho) - P(\bar{\rho})\|_{L^4}^2.$$

For the first term on the right hand side of the above inequality, we have

$$\sigma \|F\|_{L^4}^2 \leq \sigma \|F\|_{L^2}^{1/2} \|\nabla F\|_{L^2}^{3/2}$$
$$(4.30) \qquad \leq C \left(\|\nabla u\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} \right)^{1/2} \left(\int_{\mathbb{R}^3} \sigma |\dot{u}|^2 \, dx + \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |B|^2 \, dx \right)^{3/4}$$
$$\leq C \left(A_1(T_0)^{1/4} + C_0^{1/4} \right) \left(A_2(T_0)^{3/4} + E(T_0)^{3/4} \right).$$

For the second term on the right hand side of (4.29), we have

$$\sigma \|\omega\|_{L^4}^2 \leq \sigma \|\omega\|_{L^2}^{1/2} \|\nabla \omega\|_{L^2}^{3/2}$$
$$(4.31) \qquad \leq \|\nabla u\|_{L^2}^{1/2} \left(\int_{\mathbb{R}^3} \sigma |\nabla \omega|^2 \, dx \right)^{3/4}$$
$$\leq A_1(T_0)^{1/4} \left(\int_{\mathbb{R}^3} \sigma |\dot{u}|^2 \, dx + \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |B|^2 \, dx \right)^{3/4}$$
$$\leq A_1(T_0)^{1/4} \left(A_2(T_0)^{3/4} + E(T_0)^{3/4} \right).$$

For the third term on the right hand side of (4.29), we have

$$(4.32) \qquad \sigma \|P(\rho) - P(\bar{\rho})\|_{L^4}^2 \leq C \left(\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \, dx \right)^{1/2} \leq CC_0^{1/2}.$$

Substitute (4.30), (4.31), (4.32) into (4.29), we will obtain

$$(4.33) \qquad \sigma \|\nabla u\|_{L^4}^2 \leq CC_0^{1/2} + C(A_1(T_0)^{1/4} + C_0^{1/4})(A_2(T_0)^{3/4} + E(T_0)^{3/4})$$

Combining (4.28), (4.33) and (4.27), we finally have

$$(4.34) \qquad \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx \leq CA_1(T_0) \Big(C_0^{1/2} A_1(T_0)^{3/4} + C_0^{1/2}$$
$$+ (A_1(T_0)^{1/4} + C_0^{1/4})(A_2(T_0)^{3/4} + E(T_0)^{3/4}) \Big).$$

Using similar arguments as for term $\int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx$, we have

$$(4.35) \quad \int_{\mathbb{R}^3} \sigma |\nabla u|^2 |B|^2 \, dx + \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |B|^2 \, dx$$
$$\leq C A_1(T_0) \Big(C_0^{1/2} A_1(T_0)^{3/4} + A_1(T_0)^{1/4} (A_2(T_0) + E(T_0))^{3/4} \Big).$$

Summing up (4.34) and (4.35), we obtain

$$E(T_0) \leq C A_1(T_0) \Big(C_0^{1/2} A_1(T_0)^{3/4} + C_0^{1/2}$$
$$+ (A_1(T_0)^{1/4} + C_0^{1/4})(A_2(T_0)^{3/4} + E(T_0)^{3/4}) \Big)$$
$$\lesssim C_0^{1/2} A_1(T_0)^{7/4} + C_0^{1/2} A_1(T_0) + A_1(T_0)^{5/4} A_2(T_0)^{3/4}$$
$$+ A_1(T_0)^{5/4} E(T_0)^{3/4} + A_1(T_1) C_0^{1/4} A_2(T_0)^{3/4} + C_0^{1/4} A_1(T_0) E(T_0)^{3/4}.$$

Using Young's inequality, we can get

$$(4.36) \quad E(T_0) \leq C C_0^{\frac{1}{2}} A_1(T_0)^{\frac{7}{4}} + C C_0^{\frac{1}{2}} A_1(T_0) + C C_0 A_1(T_0)^4$$
$$+ C A_1(T_0)^5 + C A_1(T_0)^{\frac{5}{4}} A_2(T_0)^{\frac{3}{4}} + C C_0^{\frac{1}{4}} A_1(T_0) A_2(T_0)^{\frac{3}{4}}.$$

Through (4.36), we could easily obtain our desired result. $\qquad\square$

At this stage, we could give the estimate about term $\sigma \|\nabla^2 B\|_{L^2}^2$ as follows

$$(4.37) \quad \int_{\mathbb{R}^3} \sigma |\nabla^2 B|^2 \, dx \leq C \int_{\mathbb{R}^3} \sigma |B_t|^2 \, dx + C \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 + \sigma |\nabla u|^2 |B|^2 \, dx$$
$$\leq C A_2(T_0) + C E(T_0)$$
$$\leq C(1 + C_0^{1/2}) A_2(T_0) + C A_1(T_0)^{5/4} A_2(T_0)^{3/4}$$
$$+ C C_0^{1/4} A_1(T_0) A_2(T_0)^{3/4} + C A_1(T_0)^5 + C C_0 A_1(T_0)^4,$$

where we used Lemma 4.6.

Lemma 4.7.

$$H(T_0) \leq C P_{HC}(C_0) + C P_{HA}(A(T_0)),$$

where P_{HC} is a polynomial function with lowest order $\frac{3}{4}$ and P_{HA} is a polynomial function with lowest order $\frac{9}{8}$.

Proof. We need to estimate every term appeared in the definition of $H(T_0)$. For $\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dxdt$, we have

$$(4.38) \quad \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dxdt \leq C \int_0^{T_0} \sigma \|\nabla B\|_{L^2} \|\nabla^2 B\|_{L^2}^3 \, dt$$
$$\leq C A_1(T_0)^{1/2} \sigma^{1/2} \|\nabla^2 B\|_{L^2} \int_0^{T_0} \sigma \|\nabla^2 B\|_{L^2}^2 \, dt.$$

For the last term of the above inequality, we have

$$(4.39) \quad \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla^2 B|^2 \, dxdt \leq C \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|B_t|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dxdt$$
$$\lesssim A_1(T_0) + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \, dxdt.$$

For the last two terms in the above inequality, we firstly have

$$
\begin{aligned}
&\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^2 |u|^2 \, dx dt \\
&\leq \int_0^{T_0} \left(\int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^3} \sigma |u|^4 \, dx \right)^{1/2} dt \\
&\leq \left(\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx dt \right)^{1/2} \left(\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |u|^4 \, dx dt \right)^{1/2} \\
&\leq H(T_0)^{1/2} \left(\int_0^{T_0} \left(\int_{\mathbb{R}^3} |u|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{3/2} dt \right)^{1/2} \\
&\leq H(T_0)^{1/2} C_0^{1/4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{1/4} \left(\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx dt \right)^{1/2} \\
&\leq H(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}.
\end{aligned} \tag{4.40}
$$

Similarly, we could get

$$
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 |B|^2 \, dx dt \leq H(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}. \tag{4.41}
$$

Substitute (4.40) and (4.41) into (4.39), we will have

$$
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla^2 B|^2 \, dx dt \leq C A_1(T_0) + C H(T_0)^{1/2} C_0^{3/4} A_1(T_0)^{1/4}. \tag{4.42}
$$

Substitute estimates (4.37) and (4.42) into (4.38), we obtain

$$
\begin{aligned}
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla B|^4 \, dx dt \leq & C A_1(T_0)^{1/2} ((1 + C_0)^{1/2} A_2(T_0) \\
& + A_1(T_0)^{5/4} A_2(T_0)^{3/4} + C_0^{1/4} A_1(T_0) A_2(T_0)^{3/4} \\
& + A_1(T_0)^5 + C_0 A_1(T_0)^4)^{1/2} (A_1(T_0) \\
& + C_0^{3/4} A_1(T_0)^{1/4} H(T_0)^{1/2})
\end{aligned} \tag{4.43}
$$

For the term $\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx dt$, we have

$$
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dx dt \lesssim \int_0^{T_0} \int_{\mathbb{R}^3} \sigma (|\rho - \bar{\rho}|^4 + |F|^4 + |\omega|^4) \, dx dt. \tag{4.44}
$$

Concerning the first term on the right hand side of the above inequality, we have

$$
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\rho - \bar{\rho}|^4 \, dx dt \lesssim C_0 + \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |F|^4 \, dx dt. \tag{4.45}
$$

For the second term on the right hand side of (4.44), we have

$$
\begin{aligned}
\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |F|^4 \, dx dt &\leq \int_0^{T_0} \sigma \|F\|_{L^2} \|\nabla F\|_{L^2}^3 \, dt \\
&\leq \left(\int_{\mathbb{R}^3} |F|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^3} \sigma |\nabla F|^2 \, dx \right)^{1/2} \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla F|^2 \, dx dt.
\end{aligned} \tag{4.46}
$$

Due to

$$(4.47) \qquad \left(\int_{\mathbb{R}^3} |F|^2 \, dx\right)^{1/2} \leq \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)^{1/2} + \left(\int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 \, dx\right)^{1/2}$$
$$\leq C(C_0 + A_1(T_0))^{1/2},$$

$$(4.48) \qquad \left(\int_{\mathbb{R}^3} \sigma |\nabla F|^2 \, dx\right)^{1/2} \leq C \int_{\mathbb{R}^3} \sigma(|\dot{u}|^2 + |\nabla B|^2 |B|^2) \, dx$$
$$\leq C A_2(T_0) + C E(T_0),$$

and

$$\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla F|^2 \, dxdt \leq C \int_0^{T_0} \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla B|^2 |B|^2 \, dxdt$$

$$(4.49) \qquad \leq C A_1(T_0) + C \left(\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt\right)^{2/3} \left(\int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dxdt\right)^{1/3}$$

$$\leq C A_1(T_0) + C H(T_0)^{2/3} C_0^{1/2} A_1(T_0)^{1/6},$$

we can refine the estimate (4.47) to

$$(4.50) \qquad \int_0^{T_0} \int_{\mathbb{R}^3} \sigma |F|^4 \, dxdt \leq C(C_0 + A_1(T_0))^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0)$$
$$+ H(T_0)^{2/3} C_0^{1/2} A_1(T_0)^{1/6})$$

Concerning the last term on the right hand side of (4.44), we have

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\omega|^4 \, dxdt \leq \int_0^{T_0} \sigma \|\omega\|_{L^2} \|\omega\|_{L^6}^3 \, dt$$

$$\leq \|\omega\|_{L^2} \int_{\mathbb{R}^3} \sigma |\nabla \omega|^2 \, dx \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla \omega|^2 \, dxdt$$

$$\leq C A_1(T_0)^{1/2}(A_2(T_0) + E(T_0)) \left(\int_0^{T_0} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 \, dxdt\right.$$

$$(4.51) \qquad \left. + \int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \, dxdt\right)$$

$$\leq C A_1(T_0)^{1/2}(A_2(T_0) + E(T_0))\Big(A_1(T_0)$$

$$+ \big(\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt\big)^{2/3}\big(\int_0^{T_0} \int_{\mathbb{R}^3} |B|^6 \, dxdt\big)^{1/3}\Big)$$

$$\leq C A_1(T_0)^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0) + C_0^{1/3} H(T_0)^{2/3} A_1(T_0)^{1/3}).$$

Substitute (4.45), (4.50) and (4.51) into (4.44), we obtain that

$$\int_0^{T_0} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \, dxdt \leq C C_0 + C(C_0 + A_1(T_0))^{1/2}(A_2(T_0)$$

$$(4.52) \qquad \qquad + E(T_0))(A_1(T_0) + C_0^{1/2} A_1(T_0)^{1/6} H(T_0)^{2/3})$$

$$+ C A_1(T_0)^{1/2}(A_2(T_0) + E(T_0))(A_1(T_0)$$

$$+ C_0^{1/3} A_1(T_0)^{1/3} H(T_0)^{2/3}).$$

Concerning the term $\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt$, we have

$$
\begin{aligned}
\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^3 \, dxdt &\leq C \int_0^{T_0} \|\nabla B\|_{L^2}^{3/2} \|\nabla^2 B\|_{L^2}^{3/2} \, dt \\
&\leq \left(\int_0^{T_0} \|\nabla B\|_{L^2}^6 \, dt \right)^{1/4} \left(\int_0^{T_0} \|\nabla^2 B\|_{L^2}^2 \, dt \right)^{3/4} \\
&\leq C\|\nabla B\|_{L^2} \left(\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla B|^2 \, dxdt \right)^{1/4} \left(\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla^2 B|^2 \, dxdt \right)^{3/4} \\
&\leq C A_1(T_0)^{1/2} C_0^{1/4} \left(\int_0^{T_0} \int_{\mathbb{R}^3} |B_t|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \, dxdt \right)^{3/4} \\
&\leq C A_1(T_0)^{1/2} C_0^{1/4} \left(A_1(T_0)^{3/4} + C_0^{2/3} A_1(T_0)^{4/3} H(T_0)^{1/3} \right).
\end{aligned}
$$

(4.53)

For the term $\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt$, we have

$$
\begin{aligned}
\int_0^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt &\leq \int_0^{\sigma(T_0)} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt + \int_{\sigma(T_0)}^{T_0} \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt \\
&\leq \int_0^{\sigma(T_0)} \|\nabla u\|_{L^2}^{3/2} (\|\rho \dot{u}\|_{L^2}^{3/2} + \|P(\rho) - P(\bar{\rho})\|_{L^6}^{3/2}) \, dt \\
&\quad + \left(\int_{\sigma(T_0)}^{T_0} \int_{\mathbb{R}^3} |\nabla u|^2 dxdt \right)^{1/2} \left(\int_{\sigma(T_0)}^{T_0} \int_{\mathbb{R}^3} |\nabla u|^4 dxdt \right)^{1/2} \\
&\leq C C_0^{1/4} A_1(T_0)^{3/4} + A_1(T_0)^{3/2} + C C_0^{1/2} H(T_0)^{1/2}.
\end{aligned}
$$

(4.54)

Combining estimates (4.43), (4.52), (4.53), (4.54) and (4.36), we could finish the proof by a long but tedious calculations. □

Now, combining estimate (4.37), Lemma 4.6, Lemma 4.7 and Lemma 4.5, we will obtain

(4.55) $A_1(T_0) + A_2(T_0) \leq C P_{AC}(C_0) + C P_{AA}(A_1(T_0) + A_2(T_0))$,

where $P_{AC}(\cdot)$ is a polynomial function with lowest order $\frac{3}{4}$ and $P_{AA}(\cdot)$ is a polynomial function with lowest order $\frac{9}{8}$. Since $A_1(T_0) + A_2(T_0) \leq \epsilon_0^{1/2}$ and $C_0 = \|\rho_0 - \bar{\rho}\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|B_0\|_{H^1}^2 \leq \epsilon_0$, if $\epsilon_0 > 0$ is small enough, we have

(4.56) $A_1(T_0) + A_2(T_0) \leq C\epsilon_0^{1/4}\epsilon_0^{1/2} + C\epsilon_0^{1/16}\epsilon_0^{1/2} \leq \frac{1}{2}\epsilon_0^{1/2}$.

It remains to prove the lower and upper bound of the density. Set $\Gamma = \log(\rho)$ which satisfies

$$
(\lambda + \mu)\dot{\Gamma} + (P(\rho) - P(\bar{\rho})) = -F.
$$

For $0 < t < \sigma(T_0)$, we will have

$$
\int_0^{\sigma(T_0)} \|F\|_{L^\infty}\, dt \leq C \int_0^{\sigma(T_0)} \|F\|_{L^6}^{1/2}\|\nabla F\|_{L^6}^{1/2}\, dt
$$

$$
(4.57) \qquad \leq C \int_0^{\sigma(T_0)} (\|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2} + \|B\cdot\nabla B\|_{L^2}^{1/2})(\|\rho\dot{u}\|_{L^6}^{1/2} + \|B\cdot\nabla B\|_{L^6}^{1/2})\, dt
$$

$$
\leq C \int_0^{\sigma(T_0)} (\|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2} + \|B\cdot\nabla B\|_{L^2}^{1/2})(\|\nabla\dot{u}\|_{L^2}^{1/2} + \|B\cdot\nabla B\|_{L^6}^{1/2})\, dt.
$$

By estimate (4.56), we have

$$
(4.58) \qquad \int_0^{\sigma(T_0)} \|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2}\|\nabla\dot{u}\|_{L^2}^{1/2}\, dt \leq C\epsilon_0^{1/4},
$$

$$
\int_0^{\sigma(T_0)} \|B\cdot\nabla B\|_{L^2}^{1/2}\|\nabla\dot{u}\|_{L^2}^{1/2}\, dt
$$

$$
= \int_0^{\sigma(T_0)} t^{-1/2}(\sigma\|B\cdot\nabla B\|_{L^2}^2)^{1/4}(\sigma\|\nabla\dot{u}\|_{L^2}^2)^{1/4}\, dt
$$

$$
(4.59) \qquad \leq C\epsilon_0^{1/4} \int_0^{\sigma(T_0)} t^{-1/2}(\sigma\|\nabla\dot{u}\|_{L^2}^2)^{1/4}\, dt
$$

$$
\leq C\epsilon_0^{1/4} \left(\int_0^{\sigma(T_0)} t^{-2/3}\, dt\right)^{3/4} \left(\int_0^{\sigma(T_0)} \sigma\|\nabla\dot{u}\|_{L^2}^2\, dt\right)^{1/4}
$$

$$
\leq C\epsilon_0^{1/4}\epsilon_0^{1/8} \leq C\epsilon^{3/8},
$$

$$
\int_0^{\sigma(T_0)} \|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2}\|B\cdot\nabla B\|_{L^6}^{1/2}\, dt
$$

$$
\leq \int_0^{\sigma(T_0)} t^{-3/8}\|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/4}t^{3/8}\|\nabla^2 B\|_{L^2}^{3/4}\, dt
$$

$$
+ \int_0^{\sigma(T_0)} t^{-1/4}\|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2}t^{1/4}\|\nabla B\|_{L^4}\, dt
$$

$$
(4.60) \qquad \leq C\epsilon_0^{3/8} \left(\int_0^{\sigma(T_0)} t^{-3/5}\, dt\right)^{5/8} \left(\int_0^{\sigma(T_0)} t\|\nabla^2 B\|_{L^2}^2\, dt\right)^{3/8}
$$

$$
+ C\epsilon^{1/4} \left(\int_0^{\sigma(T_0)} t^{-1/3}\, dt\right)^{3/4} \left(\int_0^{\sigma(T_0)} t\|\nabla B\|_{L^4}^4\, dt\right)^{1/4}
$$

$$
\leq C\epsilon_0^{3/8}\epsilon_0^{1/2} + C\epsilon_0^{1/4}\epsilon_0^{9/16} \leq C\epsilon_0^{3/4},
$$

and

$$\int_0^{\sigma(T_0)} \|B \cdot \nabla B\|_{L^2}^{1/2} \|B \cdot \nabla B\|_{L^6}^{1/2} \, dt$$

$$\leq \int_0^{\sigma(T_0)} t^{-5/8} (t\|B \cdot \nabla B\|_{L^2}^2)^{1/4} \|\nabla B\|_{L^2}^{1/4} (t\|\nabla^2 B\|_{L^2}^2)^{3/8} \, dt$$

(4.61)
$$+ \int_0^{\sigma(T_0)} t^{-1/2} (t\|B \cdot \nabla B\|_{L^2}^2)^{1/4} t^{1/4} \|\nabla B\|_{L^4} \, dt$$

$$\leq C\epsilon_0^{3/8} \int_0^{\sigma(T_0)} t^{-5/8} (t\|\nabla^2 B\|_{L^2}^2)^{3/8} \, dt + C\epsilon_0^{1/4} \int_0^{\sigma(T_0)} t^{-1/2} t^{1/4} \|\nabla B\|_{L^4} \, dt$$

$$\leq C\epsilon_0^{3/8} \epsilon_0^{3/16} + C\epsilon_0^{1/4} \epsilon_0^{9/16} \leq C\epsilon_0^{9/16},$$

where we used estimate (4.37) and (4.56). Substitute (4.58), (4.59), (4.60) and (4.61) into (4.57), we obtain

(4.62)
$$\int_0^{T_0} \|F\|_{L^\infty} \, dt \leq C\epsilon_0^{1/4},$$

which implies that for $t \leq \sigma(T_0)$,

(4.63) $\inf(\log \rho_0(x)) - C\epsilon_0^{1/4} - Ct \leq \log \rho(t, x) \leq \sup(\log(\rho_0(x))) + C\epsilon_0^{1/4} + Ct.$

So we can choose ϵ_0, τ small enough such that for $t \leq \tau \leq \sigma(T_0)$,

(4.64)
$$\frac{3}{4} c_0 < \rho(t, x) < \frac{3}{2} c_0^{-1}.$$

For $\tau \leq t_1 \leq t_2 \leq T_0$, we have

$$\int_{t_1}^{t_2} \|F\|_{L^\infty} \, dt \leq C \int_{t_1}^{t_2} (\|\sqrt{\rho}\dot{u}\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^2}^{1/2})(\|\nabla\dot{u}\|_{L^2}^{1/2} + \|B \cdot \nabla B\|_{L^6}^{1/2}) \, dt$$

$$\leq C(t_2 - t_1)^{1/2} \int_{t_1}^{t_2} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|B \cdot \nabla B\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2 + \|B \cdot \nabla B\|_{L^6}^2 \, dt.$$

Now, we estimate the four integral terms on the right hand side of the above inequality. For the first two terms, we have

(4.65) $\int_{t_1}^{t_2} \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \, dt \leq A_2(T_0) \leq \epsilon_0^{1/2}, \quad \int_{t_1}^{t_2} \|\nabla\dot{u}\|_{L^2}^2 \, dt \leq A_2(T_0) \leq \epsilon_0^{1/2},$

The estimate about the third term appeared in (4.49), so we have

(4.66)
$$\int_{t_1}^{t_2} \|B \cdot \nabla B\|_{L^2}^2 \, dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 \, dx dt \leq C\epsilon_0^{1/2}.$$

For the fourth term, we have

$$\int_{t_1}^{t_2} \|B \cdot \nabla B\|_{L^6}^2 \, dt \leq \int_{t_1}^{t_2} \|\nabla B\|_{L^2}^{1/2} \|\nabla^2 B\|_{L^2}^{3/2} \, dt + \int_{t_1}^{t_2} \|\nabla B\|_{L^4}^4 \, dt$$

(4.67)
$$\leq \left(\int_{t_1}^{t_2} \|\nabla B\|_{L^2}^2 \, dt \right)^{1/4} \left(\int_{t_1}^{t_2} \|\nabla^2 B\|_{L^2}^2 \, dt \right)^{3/4} + C\epsilon_0^{1/2}$$

$$\leq C\epsilon_0^{1/4} \epsilon_0^{3/8} + C\epsilon_0^{1/2} \leq C\epsilon_0^{1/2}.$$

Combining estimates (4.65), (4.66) and (4.67), we obtain

$$(4.68) \qquad \int_{t_1}^{t_2} \|F\|_{L^\infty}\, dt \le C(t_2 - t_1)^{1/2} \epsilon_0^{1/2}$$

$$\le \eta(t_2 - t_1) + C_\eta \epsilon_0,$$

where $\eta > 0$ is a small enough positive number. With estimate (4.68) at hand, we can mimic the Navier-Stokes case [6] to obtain

$$(4.69) \qquad \frac{3}{4}c_0 < \rho(t,x) < \frac{3}{2}c_0^{-1},$$

where $0 \le t \le T_0$. With estimate (4.56) and (4.69), we can complete the proof by continuity argument. $\qquad\square$

5. Blow up Criterion and the Global Well-Posedness

In this part, we give a blow up criterion and then prove the local solution can be extended to a global one. Firstly, let me give the blow up criterion as follows.

Theorem 5.1. *For dimension $N = 3$, let (ρ, u, B) be a solution of system (1.1) satisfying*

$$\rho(0) > 0, \quad \rho - \bar{\rho} \in C([0,T]; H^2),$$
$$u, B \in C([0,T]; H^2) \cap L^2(0,T; H^3).$$

Let T^* be the maximal existence time of the solution. If $T^* < +\infty$, then it is necessary that

$$\limsup_{t\uparrow T^*} \left(\|\rho(t)\|_{L^\infty} + \|u(t)\|_{L^q} + \|B(t)\|_{L^q} \right) = +\infty,$$

for any $q \ge 6$.

Proof. We use the contradiction argument. Assume that $T^* < +\infty$ and

$$(5.1) \qquad \sup_{t\in[0,T^*)} \left(\|\rho(t)\|_{L^\infty} + \|u(t)\|_{L^q} + \|B(t)\|_{L^q} \right) = M < +\infty.$$

In what follows, we denote C to be a constant depending on T, M, $\|u_0\|_{H^2}$, $\|B_0\|_{H^2}$, $\|\rho_0 - \bar{\rho}\|_{H^2}$. Firstly, from the energy estimates, we have

$$(5.2) \qquad \int_{\mathbb{R}^3} |\rho - \bar{\rho}|^2 + \rho|u|^2 + |B|^2\, dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla B|^2\, dxdt \le C.$$

Considering both (5.1) and (5.2), for $r \in [2, \infty]$, we have

$$(5.3) \qquad \|\rho - \bar{\rho}\|_{L_T^\infty(L^r)} + \|\sqrt{\rho}u\|_{L_T^\infty(L^2)} + \|B\|_{L_T^\infty(L^2)}$$
$$+ \|\nabla u\|_{L_T^2(L^2)} + \|\nabla B\|_{L_T^2(L^2)} \le C.$$

Let $v = L^{-1}\nabla P(\rho)$ to be a solution of the following elliptic system

$$(5.4) \qquad Lv := \mu \Delta v + \lambda \nabla \mathrm{div} v = \nabla P(\rho).$$

By elliptic estimate, for $r \in [2, \infty]$, we can obtain

$$(5.5) \qquad \|\nabla v\|_{L^r} \le C\|P(\rho) - P(\bar{\rho})\|_{L^r} \le C, \quad \|\nabla^2 v\|_{L^r} \le C\|\nabla \rho\|_{L^r}.$$

Now, we introduce a new unknown $w = u - v$. We can easily know that

$$(\lambda + \mu)\Delta \mathrm{div} w = (\lambda + \mu)\Delta \mathrm{u} - \Delta(P(\rho) - P(\bar{\rho})),$$

and div$w$ is called effective viscous flux which was used by many authors[6, 14, 15, 16]. Through simple calculations, we have

$$(5.6) \qquad \rho \partial_t w - \mu \Delta w - \lambda \nabla \mathrm{div} w = \rho F + B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2),$$

where

$$(5.7) \qquad F = -u \cdot \nabla u + L^{-1} \nabla \mathrm{div}(P(\rho)u) + L^{-1} \nabla((\rho P'(\rho) - P(\rho))\mathrm{div} u).$$

Multiply (5.6) with $\partial_t w$ and integrating by parts, we have

$$(5.8) \qquad \begin{aligned} \partial_t \int_{\mathbb{R}^3} \mu |\nabla w|^2 + \lambda |\mathrm{div} w|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \rho |\partial_t w|^2 \, dx \\ \le C \|\sqrt{\rho} F\|_{L^2}^2 + C \|B \cdot \nabla B\|_{L^2}^2. \end{aligned}$$

For the first term on the right hand side of the above inequality, we can estimate as in the Navier-Stokes equations [31, 32] to obtain

$$(5.9) \qquad \|\sqrt{\rho} F\|_{L^2} \le C(1 + \|\nabla u\|_{L^2}) + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2},$$

where ϵ is a small enough positive number. Next, we need to estimate $\|B \cdot \nabla B\|_{L^2}$, $\|u \cdot \nabla B\|_{L^2}$, $\|B \cdot \nabla u\|_{L^2}$ and $\|u \cdot \nabla u\|_{L^2}$ to make our later estimate more clear. For the term $\|B \cdot \nabla B\|_{L^2}$, we have

$$\begin{aligned} \|B \cdot \nabla B\|_{L^2} &\le C \|B\|_{L^q} \|\nabla B\|_{L^{2q/(q-2)}} \\ &\le C \|\nabla B\|_{L^2} + \epsilon \|\nabla^2 B\|_{L^2} \\ &\le C \|\nabla B\|_{L^2} + \epsilon \|\partial_t B\|_{L^2} + \epsilon \|u \cdot \nabla B\|_{L^2} + \epsilon \|B \cdot \nabla u\|_{L^2}. \end{aligned}$$

Similarly, we have

$$\|u \cdot \nabla B\|_{L^2} \le C \|\nabla B\|_{L^2} + \epsilon \|\partial_t B\|_{L^2} + \epsilon \|u \cdot \nabla B\|_{L^2} + \epsilon \|B \cdot \nabla u\|_{L^2}.$$

As in the Navier-Stokes equations [31, 32], we can easily obtain

$$\begin{aligned} \|B \cdot \nabla u\|_{L^2} &\le C + C \|\nabla u\|_{L^2} + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2} \\ \|u \cdot \nabla u\|_{L^2} &\le C + C \|\nabla u\|_{L^2} + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2}. \end{aligned}$$

So summing up the above four estimates, we obtain

$$(5.10) \qquad \begin{aligned} \|B \cdot \nabla B\|_{L^2} + \|B \cdot \nabla u\|_{L^2} + \|u \cdot \nabla B\|_{L^2} + \|u \cdot \nabla u\|_{L^2} \\ \le C + C \|\nabla u\|_{L^2} + C \|\nabla B\|_{L^2} + \epsilon \|\partial_t B\|_{L^2} + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2}. \end{aligned}$$

Substitute (5.9) and (5.10) into (5.8), we will have

$$(5.11) \qquad \begin{aligned} \partial_t \int_{\mathbb{R}^3} \mu |\nabla w|^2 + \lambda |\mathrm{div} w|^2 \, dx + \int_{\mathbb{R}^3} \rho |\partial_t w|^2 \, dx \\ \le C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2}^2 + \epsilon \|\partial_t B\|_{L^2}^2. \end{aligned}$$

Multiply $\partial_t B$ to the third equation of system (1.1) and integrate by parts, we will obtain

$$(5.12) \qquad \begin{aligned} \partial_t \int_{\mathbb{R}^3} |\nabla B|^2 \, dx + \int_{\mathbb{R}^3} |\partial_t B|^2 \, dx \\ \le C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) + \epsilon \|\sqrt{\rho} \partial_t w\|_{L^2}^2 + \epsilon \|\partial_t B\|_{L^2}^2, \end{aligned}$$

where we used estimate (5.10). If ϵ is small enough, summing up (5.11) and (5.12), we will have

$$\partial_t \int_{\mathbb{R}^3} \mu|\nabla w|^2 + \lambda|\mathrm{div}w|^2 + |\nabla B|^2 \, dx + \int_{\mathbb{R}^3} \rho|\partial_t w|^2 \, dx$$
$$+ \int_{\mathbb{R}^3} |\partial_t B|^2 \, dx \leq C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2).$$

Integrate the above inequality for time variable, we will obtain

$$(5.13) \qquad \begin{aligned} \|\nabla w\|_{L_T^\infty(L^2)} + \|\nabla B\|_{L_T^\infty(L^2)} + \|\sqrt{\rho}\partial_t w\|_{L_T^2(L^2)} \\ + \|\partial_t B\|_{L_T^2(L^2)} + \|\nabla^2 w\|_{L_T^2(L^2)} \leq C. \end{aligned}$$

The above inequality combined with (5.5) implies that

$$(5.14) \qquad \|\nabla u\|_{L_T^\infty(L^2)} + \|\nabla u\|_{L_T^2(L^r)} \leq C \quad \text{for} \quad r \in [2,6].$$

For the term $\|\nabla^2 B\|_{L_T^2(L^2)}$, we will have

$$(5.15) \qquad \begin{aligned} \|\nabla^2 B\|_{L_T^2(L^2)} &\leq C\|\partial_t B\|_{L_T^2(L^2)} + \|u \cdot \nabla B\|_{L_T^2(L^2)} + \|B \cdot \nabla u\|_{L_T^2(L^2)} \\ &\leq C < +\infty, \end{aligned}$$

where we used (5.10) and (5.13). Hence, we have $\|\nabla B\|_{L_T^2(L^2)} \leq C < +\infty$ by interpolation. Now let us turn to the high order energy estimate. From the H^2 energy estimate (Lemma 4.4), we have

$$(5.16) \qquad \begin{aligned} &\int_{\mathbb{R}^3} \sigma(t)(\rho|\dot{u}|^2 + |B_t|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} \sigma(t)(|\nabla \dot{u}|^2 + |\nabla B_t|^2) \, dxdt \\ &\leq CC_0 + C\int_0^T \int_{\mathbb{R}^3} \sigma(t)|\nabla u|^4 \, dxdt + C\int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \, dxdt \\ &+ C\int_0^T \int_{\mathbb{R}^3} (|\nabla B|^2|B|^2 + |\nabla B|^2|u|^2 + |\nabla u|^2|B|^2) \, dxdt \\ &+ C\int_0^T \int_{\mathbb{R}^3} \sigma(t)(|u|^2 + |B|^2)(|B_t|^2 + |\dot{u}|^2 + |u|^2(|\nabla B|^2 + |\nabla u|^2)) \, dxdt. \end{aligned}$$

Noting that

$$(5.17) \qquad \mu\Delta w + \lambda\nabla\mathrm{div}w = \rho\dot{u} - B \cdot \nabla B + \frac{1}{2}\nabla(|B|^2),$$

elliptic estimate yields that

$$(5.18) \qquad \|\nabla^2 w\|_{L^2} \leq C\|\rho\dot{u}\|_{L^2} + C\|B \cdot \nabla B\|_{L^2}.$$

From (5.10) and (5.13), we obtain

$$(5.19) \qquad \int_0^T \int_{\mathbb{R}^3} |\nabla B|^2|B|^2 + |\nabla B|^2|u|^2 + |\nabla u|^2|B|^2 \, dxdt \leq C < +\infty.$$

Since

$$\int_0^T \int_{\mathbb{R}^3} \sigma(t)|B|^2|u|^2|\nabla B|^2 \, dxdt$$

(5.20)
$$\leq \int_0^T \left(\int_{\mathbb{R}^3} |B|^3|u|^3 \, dx\right)^{2/3} \left(\int_{\mathbb{R}^3} |\nabla B|^6 \, dx\right)^{1/3} dt$$

$$\leq \sup_{0\leq t\leq T} \left(\int_{\mathbb{R}^3} |B|^3|u|^3 \, dx\right)^{2/3} \int_0^T \|\nabla B\|_{L^6}^2 \, dt$$

$$\leq C\|B\|_{L_T^\infty(L^6)}^2 \|u\|_{L_T^\infty(L^6)}^2 \|\nabla B\|_{L_T^2(L^6)}^{1/2} \leq C < +\infty,$$

we can use similar methods to obtain

$$(5.21) \qquad \int_0^T \int_{\mathbb{R}^3} \sigma(t)(|u|^2 + |B|^2)|u|^2(|\nabla B|^2 + |\nabla u|^2) \, dxdt \leq C < +\infty.$$

Similar to the Navier-Stokes system [6], we can obtain

$$(5.22) \qquad \|\nabla u\|_{L^4}^4 \leq C\|\nabla u\|_{L^6}^2(1 + \|\sqrt{\rho}\dot{u}\|_{L^2}), \quad \|\nabla u\|_{L^3}^3 \leq C\|\nabla u\|_{L^6}^{3/2}.$$

Plugging (5.19), (5.21) and (5.22) into (5.16) and noting $\|\nabla u(t)\|_{L^6} \in L^2(0,T)$ by (5.14) and (5.15), we deduce by Gronwall's inequality that

$$(5.23) \qquad \int_{\mathbb{R}^3} \sigma(t)(\rho|\dot{u}|^2 + |B_t|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} \sigma(t)(|\nabla \dot{u}|^2 + |\nabla B_t|^2) \, dxdt \leq C.$$

From the above inequality, elliptic estimate (5.18) and Sobolev inequality, we have

$$(5.24) \qquad \|\nabla^2 w\|_{L_T^2(L^r)} \leq C, \quad \text{for} \quad r \in [2,6].$$

Using same methods as for Navier-Stokes system [6], we have

$$(5.25) \qquad \|\nabla^2 u\|_{L_T^2(L^r)} \leq C, \quad \|\nabla \rho\|_{L^r} \leq C, \quad \text{for} \quad r \in [2,6].$$

Since

$$\|\nabla^2 B\|_{L^2} \leq C(\|B_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla B\|_{L^2} + \|B\|_{L^\infty}\|\nabla u\|_{L^2})$$

$$\leq C + \frac{1}{2}\|\nabla^2 u\|_{L^2} + \frac{1}{4}\|\nabla^2 B\|_{L^2},$$

and

$$\|\nabla^2 u\|_{L^2} \leq C\|\nabla^2 w\|_{L^2} + C\|\nabla^2 v\|_{L^2}$$

$$\leq C(\|\rho\dot{u}\|_{L^2} + \|B\|_{L^\infty}\|\nabla B\|_{L^2} + \|\nabla \rho\|_{L^2})$$

$$\leq C + \frac{1}{4}\|\nabla^2 B\|_{L^2},$$

we have

$$(5.26) \qquad \|\nabla^2 u\|_{L^2} + \|\nabla^2 B\|_{L^2} \leq C.$$

By the above estimates, we easily know $\|u\|_{L^\infty} + \|B\|_{L^\infty} \leq C$ by interpolation, so we will obtain

$$(5.27) \quad \|\nabla^3 B\|_{L_T^2(L^2)} \leq C(\|\nabla B_t\|_{L_T^2(L^2)} + \|u\|_{L^\infty}\|\nabla^2 B\|_{L_T^2(L^2)}$$

$$+ \|B\|_{L^\infty}\|\nabla^2 u\|_{L_T^2(L^2)} + \|\nabla u \nabla B\|_{L_T^2(L^2)}) \leq C,$$

where we also used

$$\|\nabla u \nabla B\|_{L^2_T(L^2)} \leq \left(\int_0^T \left(\int_{\mathbb{R}^3} |\nabla u|^4 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^3} |\nabla B|^4 \, dx \right)^{1/2} dt \right)^{1/2}$$

$$\leq C\|\nabla u\|_{L^\infty_T(L^4)}\|\nabla u\|_{L^2_T(L^4)} + C\|\nabla B\|_{L^\infty_T(L^4)}\|\nabla B\|_{L^2_T(L^4)}$$

$$\leq C < +\infty.$$

Then, similar to Navier-Stokes system, we can get

$$(5.28) \qquad \int_0^T \|\nabla^3 w\|_{L^2}^2 \, dt \leq C.$$

From (5.5), (5.26), (5.27) and (5.28), we will obtain

$$\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 B(t)\|_{L^2}^2 + \int_0^T \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2 \, dt$$

$$(5.29)$$

$$\leq C + C \int_0^T \|\nabla^2 \rho(t)\|_{L^2}^2 \, dt.$$

Form the continuity equation, we have

$$(5.30) \qquad \frac{d}{dt}\|\nabla^2(\rho(t) - \bar{\rho})\|_{L^2}^2 \leq C(1 + \|\nabla u(t)\|_{L^\infty})\|\nabla^2(\rho(t) - \bar{\rho})\|_{L^2}^2$$

$$+ C\|\nabla^3 u(t)\|_{L^2}^2.$$

Summing up (5.29) and (5.30), we conclude by Gronwall's inequality that for $0 \leq t < T^*$,

$$\|\rho(t) - \bar{\rho}\|_{H^2} + \|u(t)\|_{H^2} + \|B(t)\|_{H^2} \leq C.$$

This ensures that the solution can be continued after $t = T^*$. $\qquad \square$

After we get the blow up criterion, we can give the proof about Theorem 2.8. From Theorem 2.6, Proposition 3.3 and Proposition 3.4, we obtain a solution (ρ, u, B) of (1.1) satisfying

$$\frac{c_0}{2} \leq \rho \leq 2c_0^{-1}, \quad \rho - \bar{\rho} \in \tilde{L}^\infty_T(\dot{B}^{3/p}_{p,1} \cap \dot{B}^s_{2,2}),$$

$$u, B \in \tilde{L}^\infty_T(\dot{B}^{3/p}_{p,1} \cap \dot{B}^{s-1}_{2,2}) \cap \tilde{L}^1_T(\dot{B}^{3/p+1}_{p,1} \cap \dot{B}^{s+1}_{2,2}).$$

Moreover, it holds that

$$(5.31) \qquad \|\rho - \bar{\rho}\|_{\tilde{L}^\infty_T(\dot{B}^{1-\delta}_{2,2})} \leq C(\|\rho_0 - \bar{\rho}\|_{H^{1-\delta}} + \|u_0\|_{\dot{H}^{-\delta}} + \|B_0\|_{\dot{H}^{-\delta}}),$$

$$\|(u, B)\|_{\tilde{L}^\infty_T(\dot{B}^{-\delta}_{2,2})} + \|(u, B)\|_{\tilde{L}^1_T(\dot{B}^{2-\delta}_{2,2})}$$

$$(5.32)$$

$$\leq C(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})(\|u_0\|_{\dot{H}^{-\delta}} + \|B_0\|_{\dot{H}^{-\delta}} + TP_+\|\rho_0 - \bar{\rho}\|_{H^{1-\delta}}),$$

$$(5.33) \qquad \|a\|_{\tilde{L}^\infty_T(\dot{B}^{3/p}_{p,1})} \leq 2\|a_0\|_{\dot{B}^{3/p}_{p,1}} + \frac{c_2 + c_3}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^5},$$

$$(5.34) \qquad \|u\|_{\tilde{L}^\infty_T(\dot{B}^{3/p-1}_{p,1})} + \|u\|_{\tilde{L}^1_T(\dot{B}^{3/p+1}_{p,1})} \leq \frac{c_2 + c_3}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^5},$$

$$(5.35) \qquad \|B\|_{\tilde{L}^\infty_T(\dot{B}^{3/p-1}_{p,1})} + \|B\|_{\tilde{L}^1_T(\dot{B}^{3/p+1}_{p,1})} \leq \frac{2c_3}{(1 + \|a_0\|_{\dot{B}^{3/p}_{p,1}})^2}.$$

Here and in what follows, the constant C depends only on $\lambda, \mu, c_0, \bar{\rho}, p, s$. Recall that the initial data satisfies

$$\|\rho_0 - \bar{\rho}\|_{L^2} \le c_1, \quad \|u_0\|_{\dot{B}_{p,1}^{3/p}} \le \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^5}, \quad \|B_0\|_{\dot{B}_{p,1}^{3/p}} \le \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^2},$$

$$\|u_0\|_{\dot{H}^{-\delta}} \le \frac{c_2}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^5 (1 + \|\rho_0 - \bar{\rho}\|_{H^2}^8 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})} = \tilde{c}_2,$$

$$\|B_0\|_{\dot{H}^{-\delta}} \le \frac{c_3}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^{7/3} (1 + \|\rho_0 - \bar{\rho}\|_{H^2}^8 + \|u_0\|_{L^2}^{4/3} + \|B_0\|_{L^2}^{4/3})} = \tilde{c}_3.$$

Taking c_2 small enough, by Remark **??** and (3.24), the existence time T has a lower bound

$$(5.36) \qquad T \ge \frac{c}{(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})^4 (1 + \|a_0\|_{H^2})^{12}},$$

where c is a small positive number.

Now, for any $T_1 \le T$, we have

$$(5.37) \qquad \begin{aligned} &\|(u, B)\|_{\tilde{L}_{T_1}^\infty(\dot{B}_{2,2}^{-\delta})} + \|(u, B)\|_{\tilde{L}_{T_1}^1(\dot{B}_{2,2}^{2-\delta})} \\ &\qquad \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_2 + \tilde{c}_3 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^{1-\delta}}), \end{aligned}$$

by the conditions on the solution just mentioned. For $r \in (1, 2 - \delta)$, we have

$$(5.38) \qquad \begin{aligned} \|\nabla u\|_{L_{T_1}^r(L^2)} &\le \sum_{j=-\infty}^{+\infty} 2^j \|\Delta_j u\|_{L_{T_1}^2(L^2)} \\ &\le \sum_{j \le 0} 2^j T_1^{1/r} \|\Delta_j u\|_{L_T^\infty(L^2)} + \sum_{j>0} 2^j \|\Delta_j u\|_{L_{T_1}^\infty(L^2)}^{1-1/r} \|\Delta_j u\|_{L_{T_1}^1(L^2)}^{1/r} \\ &\le C(\|u\|_{\tilde{L}_{T_1}^\infty}(\dot{B}_{2,2}^{-\delta}) + \|u\|_{\tilde{L}_{T_1}^1(\dot{B}_{2,2}^{2-\delta})}) \\ &\le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_4 + \tilde{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2}). \end{aligned}$$

Similarly, we have

$$(5.39) \qquad \|u\|_{L_{T_1}^r(L^2)} \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_4 + \tilde{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2}),$$

$$(5.40) \qquad \|\nabla B\|_{L_{T_1}^r(L^2)} \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_4 + \tilde{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2}),$$

and

$$(5.41) \qquad \|B\|_{L_{T_1}^r(L^2)} \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_4 + \tilde{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2}).$$

Then taking $r = \frac{3}{2}$, we have

$$(5.42) \qquad \|(u, B)\|_{L_{T_1}^{3/2}(H^1)} \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})(\tilde{c}_4 + \tilde{c}_5 + P_+ T_1 \|\rho_0 - \bar{\rho}\|_{H^2}).$$

Hence, there exist $t_0 \in (0, T_1)$ such that

$$(5.43) \qquad \|(u, B)(t_0)\|_{H^1} \le C(1 + \|a_0\|_{\dot{B}_{p,1}^{3/p}})\left(\frac{\tilde{c}_4 + \tilde{c}_5}{T_1^{2/3}} + P_+ T_1^{1/3} \|\rho_0 - \bar{\rho}\|_{H^2}\right).$$

For density, using simply energy estimates, we will get

$$(5.44) \qquad \|\rho(t) - \bar{\rho}\|_{L^2} \le C(1 + P_+)(c_1 + T_1^{1/2}(\|u_0\|_{L^2} + \|B_0\|_{L^2})).$$

Take T_1 as

$$T_1 = \frac{\epsilon_0^3}{(5C)^3(1+P_+)^3(1+\|\rho_0 - \bar\rho\|_{H^2}^{12} + \|(u_0, B_0)\|_{L^2}^2)(1+\|a_0\|_{\dot B_{p,1}^{3/p}})^4},$$

such that $T_1 \leq T$ and

$$C(1 + \|a_0\|_{\dot B_{p,1}^{3/p}})P_+ T_1^{1/3}\|\rho_0 - \bar\rho\|_{H^2} \leq \frac{\epsilon_0}{5},$$

$$C(1 + P_+)T_1^{1/2}(\|u_0\|_{L^2} + \|B_0\|_{L^2}) \leq \frac{\epsilon_0}{5}.$$

Then, we choose c_1, c_2 and c_3 small enough so that

$$C(1+P_+)c_1 \leq \frac{\epsilon_0}{5}, \quad \frac{C(1 + \|a_0\|_{\dot B_{p,1}^{3/p}})\tilde c_2}{T_1^{2/3}} \leq \frac{\epsilon_0}{5}, \quad \frac{C(1 + \|a_0\|_{\dot B_{p,1}^{3/p}})\tilde c_3}{T_1^{2/3}} \leq \frac{\epsilon_0}{5}.$$

Through this choice of c_1, c_2 and c_3, by (5.43) and (5.44), we have

$$\|a(t_0)\|_{L^2} + \|u(t_0)\|_{H^1} + \|B(t_0)\|_{H^1} \leq \epsilon_0.$$

Theorem 4.1 implies that

$$\frac{c_0}{2} \leq \rho \leq 2c_0^{-1}, \quad \|u(t)\|_{L_T^\infty(L^6)} + \|u(t)\|_{L_T^\infty(L^6)} \leq C.$$

So the solution can be extended to a global one by Theorem 5.1.

6. Appendix

In this section, for the reader's convenience, we firstly give the Lagrangian transform for terms appearing in the MHD system as follows

$$\widetilde{\nabla_x(|B|^2)} = J^{-1}\mathrm{div}_y(\mathrm{adj}(DX)|\tilde B|^2),$$

$$\widetilde{B \cdot \nabla_x B} = \widetilde{\mathrm{div}_x(B \otimes B)} = J^{-1}\mathrm{div}_y(\mathrm{adj}(DX)^T \tilde B \otimes \tilde B),$$

(6.1) $$\widetilde{\mathrm{div}_x uB} = J^{-1}\mathrm{div}_y(\mathrm{adj}(DX)\tilde u)\tilde B,$$

$$(\widetilde{\mathrm{div}_x u})B + \widetilde{u \cdot \nabla_x B} = \widetilde{\nabla_x(uB)} = J^{-1}\mathrm{div}_y(\mathrm{adj}(DX)\tilde u \tilde B),$$

$$\widetilde{B \cdot \nabla_x u} = \widetilde{\mathrm{div}_x(B \otimes u)} = J^{-1}\mathrm{div}_y(\mathrm{adj}(DX)^T \tilde u \otimes \tilde B).$$

Then, let us look at the following Lamé system with nonconstant coefficients

(6.2) $$\partial_t v - 2a\mathrm{div}(\mu\mathcal{D}(v)) - b\nabla(\lambda\mathrm{div}v) = f.$$

We assume that the following uniform ellipticity condition is satisfied:

(6.3) $$\alpha := \min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(a\mu)(t,x), \inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(2a\mu + b\lambda)(t,x)\right) > 0$$

Concerning this equation, we have the following lemma.

Lemma 6.1. [29, 33] *Let a, b, λ and μ be bounded functions satisfies (6.3). Assume that $a\nabla\mu$, $b\nabla\lambda$, $\mu\nabla a$ and $\lambda\nabla b$ are in $L^\infty(0, T; \dot B_p^{n/p})$ for some $1 < p < \infty$. There exist two constants η and κ such that if for some $m \in \mathbb{Z}$ we have*

(6.4) $$\min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m(2a\mu + b\lambda)(t,x), \inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m(a\mu)(t,x)\right) \geq \frac{\alpha}{2},$$

$$\|(I - S_m)(\mu\nabla a, a\nabla\mu, \lambda\nabla b, b\nabla\lambda)\|_{L_T^\infty(\dot B_p^{n/p-1})} \leq \eta\alpha,$$

then the solutions to (6.2) satisfy for all $t \in [0, T]$,

$$\|v\|_{L_t^\infty(\dot{B}_p^s)} + \alpha\|v\|_{L_t^1(\dot{B}_p^{s+2})}$$

$$\leq C\left(\|v_0\|_{\dot{B}_p^s} + \|f\|_{L_t^1(\dot{B}_p^s)}\right)\exp\left(\frac{C}{\alpha}\int_0^t \|S_m(\mu\nabla a, a\nabla\mu, \lambda\nabla b, b\nabla\lambda)\|_{\dot{B}_p^{n/p}}^2\,d\tau\right),$$

whenever $-\min(n/p, n/p') < s \leq n/p - 1$ *is satisfied.*

Lemma 6.2. [34] *Let* $p \in [1, +\infty]$, $1 \leq \rho_2 \leq \rho_1 \leq +\infty$, *and let* v *solve*

$$\begin{cases} \partial_t v - \nu\triangle v = f, \\ v|_{t=0} = v_0. \end{cases}$$

Denote $\rho_1' = (1 + 1/\rho_1 - 1/\rho_2)^{-1}$. *Then there exist two positive constants* c *and* C *depending only on* N *and such that*

$$\|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_p^{s+2/\rho_1})} \leq \left(\sum_{q\in\mathbb{Z}} 2^{qs}\|\Delta_q v_0\|_{L^p}\left(\frac{1 - e^{-c\nu T2^{2q}\rho_1}}{c\nu\rho_1}\right)^{1/\rho_1}\right.$$

$$\left.\sum_{q\in\mathbb{Z}} 2^{q(s-2+2/\rho_2)}\|\Delta_q f\|_{L_T^{\rho_2}(L^p)}\left(\frac{1 - e^{-c\nu T2^{2q}\rho_1'}}{c\nu\rho_1'}\right)^{1/\rho_1'}\right).$$

In particular, we have

$$\|v\|_{\tilde{L}_T^{\rho_1}(\dot{B}_p^{s+2/\rho_1})} \leq \frac{C}{\nu^{1/\rho_1}}\|v_0\|_{\dot{B}_p^s} + \frac{C}{\nu^{1/\rho_1'}}\|f\|_{\tilde{L}_T^{\rho_2}(\dot{B}_p^{s-2+2/\rho_2})}.$$

Moreover u *belongs to* $C([0, T]; \dot{B}_p^s)$.

Acknowledgments

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